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The Newton Polytope of the Optimal Differential Operator for an Algebraic Curve

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We investigate the linear differential operator with polynomial coefficients whose space of holomorphic solutions is spanned by all the branches of a function defined by a generic algebraic curve. The main result is a description of the coefficients of this operator in terms of their Newton polytopes.

Keywords: algebraic function, minimal differential operator, Newton polytope.

Introduction

To find relations satisfied by a given special function is a difficult and important problem in the theory of special functions of mathematical physics. The relations in question can involve derivatives, integrals, finite differences etc. Knowing a global relation for a special function that is defined locally (e.g. by means of a series converging in a neighbourhood of a point) allows one to deduce global properties of that function. From this point of view, linear differential equations with polynomial coefficients are of particular interest. One of the reasons for this is the difficult problem of computing the analytic continuation along a given path of a locally defined special function. By identifying such a function with a solution to a system of linear differential equations with polynomial coefficients which does not have any "extra solutions" one can use standard techniques for investigating the analytic continuation of the function under study (see, for instance, [14]). Here by "extra solutions" we mean the solutions which are not branches of the function under study, that is, which cannot be obtained from it by means of analytic continuation. Observe that every germ of a (multivalued) analytic function satisfies a relation with entire (in particular, polynomial) coefficients provided that this relation is valid for one of its germs in a neighbourhood of some fixed nonsingular point.

The culmination of this approach is the Wilf-Zeilberger algorithmic proof theory (see [18] and [19]) based on holonomic systems of equations. In the present paper we thoroughly investigate the special case when the function under study is algebraic and the holonomic system consists of a single ordinary linear homogeneous differential equation with polynomial coefficients. Despite all simplicity, this setup leads to formidable computational challenges.

The 21st problem in the Hilbert list was solved in 1989 by A.A. Bolibrukh who proved that it is in general not possible to construct a linear fuchsian system of differential equations with

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a prescribed monodromy group (see [1]). However, the problem of effective computation of a system of differential equations (and, in particular, of a single differential equation) with a prescribed branching of solutions (whenever this is possible) remains open and is in the focus of intensive research, see [4, 6, 9]. The computer algebra system *Magma* has a built-in command *DifferentialOperator* for finding such operators (see [5]). Another powerful tool for finding linear differential equations (both homogeneous and inhomogeneous) for algebraic functions is the *gfun* package developed by B. Salvy et. al. (see [4]).

In the present paper we describe an algorithm which allows one to compute annihilating operators for an essentially larger class of algebraic functions (see Examples 4 (4), 9, 10 and 11). We also provide a combinatorial characterization of the coefficients of the optimal annihilating operators in terms of their Newton polytopes.

It is well-known that an ordinary linear differential equation with a prescribed solution space can be found by means of the wronskian of a basis of this space. However, from the computational point of view, the wronskian-based representation of the differential equation for an analytic function (which is, in general, defined only locally) is merely a nonconstructive existence theorem. There are three main reasons for this. First, to form the wronskian, one needs to choose a basis in the space of germs of the given function at a nonsingular point. This requires computing the analytic continuation of the given function along any path, which is, in general, a difficult problem. Secondly, to evaluate a determinant containing high-order derivatives of a given special function is a task of a great computational complexity. Finally, extracting the polynomial coefficients of the desired differential operator out of the obtained combination of algebraic functions requires the full use of modern methods of computer algebra. For instance, to compute the differential operator for the roots of the generic monic cubic by means of the wronskian is already a challenge (see example in Sec. 5 in [11]). In the general case, the wronskian-based construction is not suitable for computation since no effective means of simplifying expressions which contain high-order derivatives of special functions are presently known.

The present paper provides an algorithm for constructing the optimal (that is, of the smallest possible order) linear homogeneous differential equation with polynomial coefficients for a univariate algebraic function $y = y(x)$ implicitly defined by the equation

$$y^m + a_1 y^{m_1} + \dots + a_n y^{m_n} + x = 0. \tag{1}$$

The proposed method is a development of the ideas of the work [11]. It allows one to reduce the problem of computing the annihilating operator for an algebraic function to the problem of finding a basis in the syzygy module of an ideal in the ring of multivariate polynomials. The presented algorithm differs from other methods (both recent and classical, see [4, 6, 9]) in its primary field of application (it deals with generic algebraic equations), in the underlying concept (holonomic systems of partial differential equations and noncommutative elimination) and the complexity of differential operators that it can efficiently produce. The capabilities and limitations of the proposed algorithm are summarized in Table 1.

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1. Annihilating operators for solutions to holonomic systems of differential equations

In what follows we will denote by \mathcal{D}_n the Weyl algebra of differential operators with polynomial coefficients in n variables $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. This algebra is generated by the operators $x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}$ satisfying the relations $\partial_{x_i} \circ x_j - x_j \circ \partial_{x_i} = \delta_{ij}$. Here " \circ " denotes the

composition of differential operators. The Weyl algebra is simple (see Chapter 1 in [2]). When speaking about ideals in the Weyl algebra, we will always mean its left ideals.

The following basic statement is well-known but not easy to find in the literature in the following explicit form. It can be deduced from Theorem 2 in [17]. It also follows from Theorem 1.4.12, Proposition 1.4.9, and Lemma 2.2.3 in [15].

Proposition 1. *For any holonomic left ideal $I \subset \mathcal{D}_n$ and any $i \in \{1, \dots, n\}$ there exists a nonzero operator $P_i \in I$, all of whose derivatives are with respect to the variable x_i , that is, an operator of the form*

$$P_i = \sum_{j=1}^{N_i} a_{ij}(x_1, \dots, x_n) \partial_{x_i}^j.$$

The following statement is a consequence of the results in [10] and [12]. It shows that algebraic functions defined by generic algebraic curves are annihilated by holonomic ideals in \mathcal{D}_n .

Theorem 2. *Any germ of the algebraic function $y(x_0, x_1, \dots, x_n)$ implicitly defined by the relation*

$$x_n y^n + x_{n-1} y^{n-1} + \dots + x_1 y + x_0 = 0, \tag{2}$$

satisfies the holonomic system of differential equations

$$\begin{aligned} \partial_{x_i} \partial_{x_j} y &= \partial_{x_k} \partial_{x_l} y, \text{ whenever } i + j = k + l, \\ \sum_{i=0}^n i x_i \partial_{x_i} y &= -y \quad \text{and} \quad \sum_{i=0}^n x_i \partial_{x_i} y = 0. \end{aligned} \tag{3}$$

Conversely, any holomorphic solution of (3) defined locally in a neighbourhood of a nonsingular point is a linear combination of germs of the function $y(x_0, x_1, \dots, x_n)$ at this point.

The system of differential equations (3) is a special instance of the Gelfand-Kapranov-Zelevinsky hypergeometric system introduced in [10]. Its "dehomogenized" version for an algebraic curve with affine parameters was investigated by Mellin in [13].

Recall that the *Nilsson class* comprises (multi-valued) analytic functions of several complex variables which have finite determination and moderate growth in arbitrary neighborhood of any of their singularities (see 4.1.12 in [3]). Here by the determination of a multi-valued analytic function we mean the number of its linearly independent germs in a neighborhood of a generic point in its domain of definition. The determination of an analytic function of one complex variable which lies in the Nilsson class and has finitely many singularities in \mathbb{C} coincides with the smallest possible order of an ordinary linear homogeneous differential equation with polynomial coefficients satisfied by this function.

Theorem 2 together with Proposition 1 imply the existence of a linear differential operator with polynomial coefficients whose space of local holomorphic solutions at a nonsingular point is spanned by the roots of the generic algebraic equation (2) and all of whose derivatives are with respect to x_0 . This operator is defined uniquely up to a sign. We will say that this operator is *optimal* for the given generic algebraic curve.

Example 3. Consider the algebraic function $y(x_0, x_1, x_2)$ defined as the solution to the quadratic equation $x_2 y^2 + x_1 y + x_0 = 0$. By Theorem 2, any of its branches lies in the kernel of any operator in the ideal J with the generators

$$A = \partial_{x_0} \partial_{x_2} - \partial_{x_1}^2, \quad B = x_1 \partial_{x_1} + 2x_2 \partial_{x_2} + 1, \quad C = x_0 \partial_{x_0} + x_1 \partial_{x_1} + x_2 \partial_{x_2}.$$

Since the determination of the function $y(x_0, x_1, x_2)$ equals 2, Proposition 1 yields the existence of a second order differential operator $P \in J$, all of whose derivatives are with respect to x_0 .

Using the notation $\theta_i = x_i \partial_{x_i}$, we can write the expansion of this operator with respect to the basis of J in the form

$$\begin{aligned} P &= x_0 x_1^2 x_2 A - ((x_1^2 - 2x_0 x_2) \theta_0 + x_0 x_2 \theta_1) B + ((x_1^2 - 4x_0 x_2) \theta_0 + 2x_0 x_2 \theta_1) C = \\ &= x_0^2 ((x_1^2 - 4x_0 x_2) \partial_{x_0}^2 - 2x_2 \partial_{x_0}). \end{aligned}$$

Of course, this optimal differential operator is only a monomial multiple of the wronskian of the roots of the initial algebraic equation.

In the next section we describe the algorithm for computing the optimal annihilating operator for an arbitrary algebraic function satisfying an equation of the form (1). This will, in particular, perform the noncommutative elimination of all the derivatives except for ∂_{x_i} in the holonomic ideal (3) by means of methods of commutative algebra only.

2. Computing the annihilating operator for a given algebraic function

We begin by computing the determinations of some elementary functions and the corresponding differential equations.

Example 4. (1) Any rational function $f = p(x)/q(x)$, where $p(x), q(x) \in \mathbb{C}[x]$, has determination 1 and satisfies the first-order differential equation $pqf' = (p'q - pq')f$.

(2) The function $f = x^a$ also has determination 1 for any $a \in \mathbb{C}$ since its analytic continuation $e^{2\pi i a} f$ around the only finite singularity $x = 0$ is proportional to f . It satisfies the differential equation $xf' = af$.

(3) The function $f = \ln x$ has determination 2, since its analytic continuation along any path can be written in the form $\ln x + 2\pi ki$, $k \in \mathbb{Z}$. Thus any germ of f at a nonsingular point lies in the two-dimensional linear space with the basis $\{1, \ln x\}$. The second-order differential equation with polynomial coefficients satisfied by f has the form $xf'' + f' = 0$.

(4) The algebraic function $y = y(x)$ implicitly defined by the relation $y^5 + ay + x = 0$ has determination 4 (see Theorem 5 below) and satisfies the differential equation $(256a^5 + 3125x^4)y^{(4)} + 31250x^3y^{(3)} + 73125x^2y^{(2)} + 31875xy' - 1155y = 0$.

(5) Finally, the function $1/\ln x$ has infinite determination since its germs $\{1/(\ln x + 2\pi ki)\}_{k \in \mathbb{Z}}$ are linearly independent. This implies, in particular, that this function does not satisfy any linear homogeneous differential equation with polynomial coefficients.

In the present section we describe an algorithm for computing the optimal annihilating operator for the roots of a generic algebraic equation with symbolic coefficients, that is, an equation of the form (1). The roots of the equation $a_0 y^m + a_1 y^{m_1} + a_2 y^{m_2} + \dots + a_n y^{m_n} + a_{n+1} = 0$ (regarded as functions of $a = (a_0, \dots, a_{n+1})$) satisfy the holonomic A -hypergeometric system with the vector of parameters $(0, -1)$ (see [16]), where

$$A := \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ m & m_1 & \dots & m_n & 0 \end{pmatrix}.$$

Namely, it is the left ideal in the Weyl algebra $\mathbb{C}[a_0, \dots, a_{n+1}, \partial_0, \dots, \partial_{n+1}]$ generated by the toric operators $\partial^u - \partial^v$ for $u, v \in \mathbb{N}^{n+2}$ with $A \cdot u = A \cdot v$, and the Euler operators $\sum_{j=0}^{n+1} a_j \partial_j$ and

$ma_0 + \sum_{j=1}^n m_j a_j \partial_j + 1$. Thus by Proposition 1, there always exists a linear differential operator

with polynomial coefficients in a_0, \dots, a_{n+1} , and all of whose derivatives are with respect to a_{n+1} . Setting $a_0 = 1$ and $a_{n+1} = x$ we obtain the annihilating operator for the solutions of (1). Using noncommutative elimination theory, one can compute this operator in a way similar to that in Example 3. In the special case of a trinomial equation (that is, for $n = 1$) the desired operator is a right factor of the Mellin differential operator found in [13].

The following theorem gives the order of the annihilating operator.

Theorem 5 (E. Cattani, C. D’Andrea, A. Dickenstein [7]). *The number of linearly independent (over the field of complex numbers) germs of the solutions to the equation (1) at a generic point $x \in \mathbb{C}$ and for generic values of the parameters $(a_1, \dots, a_n) \in \mathbb{C}^n$ is given by*

$$R(m, m_1, \dots, m_n) = \begin{cases} m - 1 + \left\lfloor \frac{m_1}{m - 1} \right\rfloor, & \text{if } \text{GCD}(m, m_1, \dots, m_n) = 1, \\ \frac{m}{\text{GCD}(m, m_1, \dots, m_n)}, & \text{if } \text{GCD}(m, m_1, \dots, m_n) > 1. \end{cases}$$

Here $\lfloor \cdot \rfloor$ denotes the integer part of a real number.

The following theorem is the foundation of our algorithm for computing optimal annihilating operators.

Theorem 6. *Let $s_i = s_i(x, a_1, \dots, a_n)$, $i = 1, \dots, m$ be the roots of the algebraic equation (1) and denote $P(t) = \prod_{i=1}^m (t - s_i)$. For every $k = 1, \dots, m$ we define the ideal I_k in the polynomial ring with $m + n + 1$ variables $\mathbb{C}[s_1, \dots, s_m, a_1, \dots, a_n, x]$ to be*

$$\left((-1)^\ell (\ell - 1)! \left(\prod_{i \neq k} (s_k - s_i)^{2m-1} \right) \text{res}_{t=s_k} \frac{1}{P(t)^\ell}, \ell = 1, \dots, R(m, m_1, \dots, m_n) \right).$$

The vector of polynomial coefficients of the optimal annihilating operator for the algebraic function defined by (1) lies in the following syzygy module of the quotient of the ideal I_k with respect to the Vieta relations:

$$\text{Syz} \left(I_k / \left(\begin{aligned} & \mathcal{S}_{m-m_j}(s_1, \dots, s_m) - (-1)^{m-m_j} a_j, \quad j = 1, \dots, n; \\ & \mathcal{S}_{m-k}(s_1, \dots, s_m), \quad \text{for } k \notin \{0, m_1, m_2, \dots, m_n\}; \\ & \mathcal{S}_m(s_1, \dots, s_m) - (-1)^m x \end{aligned} \right) \right). \tag{4}$$

Here $\mathcal{S}_j(s_1, \dots, s_m)$ is the elementary symmetric polynomial of order j in the variables (s_1, \dots, s_m) . In the sequel we will denote the ideal generated by the Vieta relations by \mathcal{V} .

Proof. Let $x \in \mathbb{C}$ be a point outside of the zero locus of the discriminant of the left-hand side in (1). Let $(a_1, \dots, a_n) \in \mathbb{C}^n$ be a generic vector of parameters and let $y_k(x, a_1, \dots, a_n)$ denote the k -th branch of the solution $y(x, a_1, \dots, a_n)$ to the algebraic equation (1). Let us now denote by D the differential operator $D = \partial_{s_1} \dots \partial_{s_m}$. Using the well-known contour integral representation for a solution to a univariate algebraic equation (see Section 5 in [11]) we conclude that the generators of the ideal I_k are polynomial multiples of the derivatives of the solution to (1):

$$\begin{aligned} \partial_x^\ell y_k(x, a_1, \dots, a_n) &= (-1)^\ell (\ell - 1)! \text{res}_{t=s_k} \frac{1}{P(t)^\ell} = \frac{(-1)^\ell}{((\ell - 1)!)^{m-1}} D^{\ell-1} \left(\text{res}_{t=s_k} \frac{1}{P(t)} \right) = \\ &= \frac{(-1)^\ell}{((\ell - 1)!)^{m-1}} D^{\ell-1} \left(\frac{1}{(s_k - s_1) \dots [k] \dots (s_k - s_n)} \right) = (-1)^\ell \partial_{s_k}^{\ell-1} \frac{1}{((s_k - s_1) \dots [k] \dots (s_k - s_n))^\ell} \end{aligned}$$

$$= - \sum_{i_1+\dots+[k]\dots+i_n=\ell-1} \frac{(\ell+i_1-1)! \dots [k] \dots (\ell+i_n-1)!}{((\ell-1)!)^{m-2} i_1! \dots [k] \dots i_m! (s_k-s_1)^{\ell+i_1} \dots [k] \dots (s_k-s_n)^{\ell+i_n}}.$$

This shows that the generators of the ideal I_k are indeed elements of the ring $\mathbb{C}[s_1, \dots, s_m, a_1, \dots, a_n, x]$. By Theorem 5 the determination of the solution to (1) equals $R(m, m_1, \dots, m_n)$. Thus by Theorem 2 and Proposition 1 there exists a linear differential operator with polynomial coefficients (in x, a_1, \dots, a_n) all of whose derivatives are with respect to x and whose space of holomorphic solutions at a generic point is spanned by the branches of $y(x, a_1, \dots, a_n)$. For the sake of computational efficiency we factor out the Vieta relations. This increases the number of variables involved in the generators of the ideal but decreases their degrees. The desired differential operator is a relation between the derivatives of $y(x, a_1, \dots, a_n)$ with polynomial (and thus single-valued) coefficients. By the conservation principle for analytic functions the same relation must be satisfied by any of the germs of $y(x, a_1, \dots, a_n)$ at a nonsingular point. Thus the coefficients of this relation lie in the syzygy module (4). \square

Observe that the elements of the syzygy module (4) are polynomial vectors whose entries in general depend on all of the variables $s_1, \dots, s_m, a_1, \dots, a_n, x$. The proof of Theorem 6 implies that there exists an element of (4) whose entries only depend on a_1, \dots, a_n, x . It can be found by means of the following algorithm.

Algorithm 7. The actual computation of annihilating operators for algebraic functions was organized as follows:

1. Compute the basis of the ideal I_1 defined in Theorem 6.
2. Using the lexicographic order of the variables s_1, \dots, s_m compute the Gröbner basis of the ideal \mathcal{V} defined by the Vieta relations (as defined in Theorem 6).
3. Perform polynomial reduction of the generators of the ideal I_1 by means of the Gröbner basis of the ideal \mathcal{V} . That is, at this step, we use the Vieta relations as much as possible in order to simplify the generators of \mathcal{V} .
4. Factorize the obtained family of polynomials. The result has a very specific structure: it is a family of polynomials in $\mathbb{C}[s_1, \dots, s_m, a_1, \dots, a_n, x]$ whose elements are symmetric with respect to s_2, \dots, s_m . Using the Gröbner basis of the ideal \mathcal{V} , reduce them to polynomials in $\mathbb{C}[s_1, a_1, \dots, a_n, x]$. Let us denote this family of polynomials by $\mathcal{R}_1, \dots, \mathcal{R}_m$.
5. Any $\mathbb{C}[a_1, \dots, a_n, x]$ -linear relation for the family of polynomials $\mathcal{R}_1, \dots, \mathcal{R}_m$ transforms into a linear system of algebraic equations over the field of rational functions in the variables a_1, \dots, a_n, x . Proposition 1 and Theorem 5 yield the existence of an at least one-dimensional \mathbb{C} -vector space of solutions to this system of linear equations. Finding a basis in this space and clearing the denominators, we obtain the desired polynomial coefficients of the optimal annihilating operator for the initial algebraic function. \square

Example 8. The linear space spanned by the roots of the algebraic equation

$$y^5 + 2y^4 - 3y^3 + y^2 + 5y + x = 0 \tag{5}$$

(in a neighbourhood of a point where the discriminant of this equation does not vanish) coincides with the linear space of holomorphic solutions to the differential equation

$$\begin{aligned} & (-43728190560 + 795819153x - 53446888x^2 + 56028x^3) \times \\ & \times (-1585575 + 71982x + 281583x^2 + 81342x^3 + 3125x^4) y^{(5)} + \\ & + 15(-650327879439783 - 5747872136026563x - 2400588229818366x^2 - \\ & - 91559102743545x^3 - 304019551433x^4 - 131338505212x^5 + 128397500x^6) y^{(4)} + \\ & + 60(-1821690090417321 - 1560609625036728x - 98711280942848x^2 + \\ & + 721492325057x^3 - 103787727624x^4 + 91045500x^5) y^{(3)} + \\ & + 180(-282046871305467 - 38794189010031x + 478890241959x^2 - \\ & - 28458003540x^3 + 21944300x^4) y'' + \\ & + 720(-1756652589603 + 23053844253x - 812236372x^2 + 522928x^3) y' = 0. \end{aligned}$$

The following example provides a fundamental system of solutions to a fifth-order linear differential equation with polynomial coefficients.

Example 9. For any $a \in \mathbb{C}^*$ a basis in the space of holomorphic solutions to the differential equation

$$\begin{aligned} & ((256/5) a^5 x^3 + 625 x^4) y^{(5)} + (384 a^5 x^2 + 6875 x^3) y^{(4)} + \\ & + (624 a^5 x + 19500 x^2) y^{(3)} + (168 a^5 + 14100 x) y'' + 1344 y' = 0 \end{aligned}$$

in a neighbourhood of a generic point $x \in \mathbb{C}$ is given by the roots of the algebraic equation $y^5 + ay^4 + x = 0$.

3. Software, hardware and examples

Most of the examples in this paper were computed by means of a *Mathematica 7.0* package developed by the authors and run on an Intel Core(TM) Duo CPU clocked at 2.00GHz. The bottleneck of the algorithm is computing the syzygy module of an ideal in a ring of polynomials in several variables. In some cases, we have used *Singular* for this.

Example 10. *Generic quintic.* One of the goals of the research presented in this paper was to provide a computationally efficient extension of the results of Section 5 in [11] beyond the class of algebraic equations with elementary solutions. In this example, we demonstrate the efficiency of the described approach by means of the generic monic quintic

$$y^5 + a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + x = 0. \tag{6}$$

Computing the annihilating operator for the solutions of this equation has turned out to be a task of considerable computational complexity. The full output of the algorithm is a vector of five polynomials with 4306 monomials in total and is too large to display. The degrees of these polynomials with respect to the variables a_1, \dots, a_4, x are 15, 20, 21, 22, 23. The leading coefficient of the annihilating operator has degree 7 with respect to x and splits into the product of two factors. One of them is the discriminant of (6) while the other is a polynomial of total degree 15 with 264 terms. The largest of the numeric coefficients in the annihilating operator for the generic monic quintic equals 2739594525000.

Example 11. *A monic tetranomial with generic coefficients.* By Theorem 5 the determination of a solution to the algebraic equation

$$y^6 + ay^2 + by + x = 0 \tag{7}$$

equals five. The roots of (7) at a generic point $x \in \mathbb{C}$ span the space of holomorphic solutions of the following fifth order linear differential operator with polynomial coefficients:

$$\begin{aligned} & (-255664128 a^{10} + 395740000 a^5 b^4 + 1599609375 b^8 + 148780800 a^6 b^2 x + 2859609375 ab^6 x - \\ & - 499654656 a^7 x^2 - 1573425000 a^2 b^4 x^2 + 1051704000 a^3 b^2 x^3 + 16796160 a^4 x^4) \times \\ & \times (256 a^5 b^2 + 3125 b^6 - 1024 a^6 x - 22500 ab^4 x + 43200 a^2 b^2 x^2 - 13824 a^3 x^3 - 46656 x^5) \frac{d^5}{dx^5} + \\ & + (916300234752 a^{16} + 18677130035200 a^{11} b^4 - 38094525000000 a^6 b^8 - 134905517578125 ab^{12} - \\ & - 77437887971328 a^{12} b^2 x + 107910691200000 a^7 b^6 x + 332702753906250 a^2 b^{10} x + \\ & + 37877629059072 a^{13} x^2 - 406052352000 a^8 b^4 x^2 + 552267618750000 a^3 b^8 x^2 - \\ & - 128020162314240 a^9 b^2 x^3 - 727448202000000 a^4 b^6 x^3 + 267464667561984 a^{10} x^4 + \\ & + 43693344000000 a^5 b^4 x^4 - 1306049062500000 b^8 x^4 - 221398918963200 a^6 b^2 x^5 - \\ & - 2201395927500000 ab^6 x^5 + 359825022517248 a^7 x^6 + 1137850610400000 a^2 b^4 x^6 - \\ & - 711490376448000 a^3 b^2 x^7 - 10579162152960 a^4 x^8) \frac{d^4}{dx^4} + \end{aligned}$$

$$\begin{aligned}
 &+30(-3264411795456 a^{12}b^2 + 5653930240000 a^7b^6 + 24016248046875 a^2b^{10} + \\
 &+3132022849536 a^{13}x - 5896377011200 a^8b^4x - 145456875000 a^3b^8x - \\
 &-2678330105856 a^9b^2x^2 - 31473123300000 a^4b^6x^2 + 38750783864832 a^{10}x^3 - \\
 &-41250841920000 a^5b^4x^3 - 221475515625000 b^8x^3 - 21927996518400 a^6b^2x^4 - \\
 &-344319609375000 ab^6x^4 + 52068310990848 a^7x^5 + 163366013160000 a^2b^4x^5 - \\
 &-95796142732800 a^3b^2x^6 - 1311148560384 a^4x^7) \frac{d^3}{dx^3} - \\
 &-120(-441539395584 a^{13} + 907546908800 a^8b^4 + 3520234921875 a^3b^8 - \\
 &-122881784832 a^9b^2x + 2644039462500 a^4b^6x - 14126136238080 a^{10}x^2 + \\
 &+20651647680000 a^5b^4x^2 + 88754326171875 b^8x^2 + 4397772787200 a^6b^2x^3 + \\
 &+122488079765625 ab^6x^3 - 16700798121984 a^7x^4 - 51772358925000 a^2b^4x^4 + \\
 &+28447624699200 a^3b^2x^5 + 352185242112 a^4x^6) \frac{d^2}{dx^2} - \\
 &-2520(4704614400 a^9b^2 + 4079375000 a^4b^6 - 219268374528 a^{10}x + \\
 &+350501380000 a^5b^4x + 1464693750000 b^8x + 5043513600 a^6b^2x^2 + \\
 &+1670568609375 ab^6x^2 - 196541448192 a^7x^3 - 597319515000 a^2b^4x^3 + \\
 &+304660958400 a^3b^2x^4 + 3302125056 a^4x^5) \frac{d}{dx} + \\
 &+5040(-2372960256 a^{10} + 4421840000 a^5b^4 + 18235546875 b^8 - 1113523200 a^6b^2x + \\
 &+14120437500 ab^6x - 1226244096 a^7x^2 - 3581820000 a^2b^4x^2 + \\
 &+1587859200 a^3b^2x^3 + 13436928 a^4x^4).
 \end{aligned}$$

Recall that the Newton polytope of a multivariate Laurent polynomial

$$f(x_1, \dots, x_n) = \sum_{\alpha=(\alpha_1, \dots, \alpha_n) \in A \subset \mathbb{Z}^n} c_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

supported in a finite set A is defined to be the convex hull of A . In what follows, we will not make any difference between two polytopes that only differ by a translation with respect to a real vector.

It turns out to be convenient to associate a single integer convex polytope with the optimal differential operator defined by an algebraic curve rather than consider the Newton polytope of each of its polynomial coefficients individually. Throughout the rest of the paper, we will be using the following definition.

Definition 12. By the *Newton polytope of the optimal differential operator* defined by a generic algebraic curve we will mean the convex hull of the exponents of all of the monomials that are present in its coefficients.

The following theorem describes the structure of the Newton polytope of the optimal differential operator for an algebraic curve. The idea of its proof has been suggested by L. Matusevich to whom the authors are very grateful.

Theorem 13. Let $\sum_{k=\ell}^d p_k(a_1, \dots, a_n, x) d_x^k$ be the optimal annihilating operator for the algebraic function defined by the relation $P(x, y) := y^m + a_1 y^{m_1} + \dots + a_n y^{m_n} + x = 0$. Then the exponent vector (v_1, \dots, v_{n+1}) of any of the monomials that appear in $p_k(a_1, \dots, a_n, x)$ lies in the hyperplane

$$(m - m_1)v_1 + \dots + (m - m_n)v_n + mv_{n+1} = \text{const.}$$

Proof. Recall that any germ of the algebraic equation $y(a_0, \dots, a_{n+1})$ defined by $a_0 y^m + a_1 y^{m_1} + \dots + a_n y^{m_n} + a_{n+1} = 0$ satisfies the A -hypegeometric system of partial differential equations $H_A(\beta)$, where $\beta = (-1, 0)$ and the matrix A is given by

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & m_n & m_{n-1} & \dots & m_1 & m \end{pmatrix}.$$

This yields, in particular, that the function $y(a_0, \dots, a_{n+1})$ is A -homogeneous.

To simplify the notation, we denote $x = a_{n+1}$ and introduce the grading on the ring of ordinary differential operators $\mathbb{C}[a_1, \dots, a_n, x, \frac{d}{dx}]$ by setting $\deg a_i = m - m_i$, for $i = 1, \dots, n$, $\deg a_0 = 0$ and $\deg x = -\deg(\frac{d}{dx}) = m$. The annihilating ideal of the function $y(a_0, \dots, a_n, x)$ is homogeneous with respect to this grading. It follows from Proposition 1 and Theorem 2 that this ideal is principal. In fact, it is generated by the optimal linear differential annihilating operator for the function $y(a_0, \dots, a_n, x)$ with its order being equal to the determination of this function. Therefore, the optimal annihilating operator is also homogeneous with respect to the introduced grading.

Observe that since $\deg a_0 = 0$, passing over to the monic equation (that is, setting $a_0 = 1$) has no effect on the grading. Thus the restriction of the optimal operator to $a_0 = 1$ is still homogeneous with respect to the grading defined above. Applying this homogeneous linear operator to the homogeneous function $y(a_1, \dots, a_n, x)$ which lies in its kernel, we conclude, that every coefficient of the optimal operator is a homogeneous polynomial with respect to the introduced grading.

Denoting by d_ℓ the degree of the polynomial $p_\ell(a_1, \dots, a_n, x)$, with respect to the grading defined above, we conclude that the degree of $p_k(a_1, \dots, a_n, x)$ equals $d_0 + k \cdot m$. This is exactly the conclusion of the theorem. \square

Intensive experiments suggest that there is an intrinsic relation between the two extreme coefficients in the optimal annihilating differential operator for an algebraic function. As we have seen in several examples before, the leading coefficient in the annihilating operator is typically given by the product of the discriminant of the defining algebraic equation and some other factor which has no apparent relation to the initial algebraic equation. The following conjecture summarizes the results of our computer experiments with the structure of the Newton polytope of this polynomial factor.

Conjecture 14. Let $\sum_{k=\ell}^d p_k(a_1, \dots, a_n, x)d_x^k$ be the optimal annihilating operator for the algebraic function defined by the relation $P(x, y) := y^m + a_1y^{m_1} + \dots + a_ny^{m_n} + x = 0$. Denote by $\mathfrak{D}(a_1, \dots, a_n, x)$ the discriminant of $P(x, y)$ computed with respect to y . Then the polynomials

$$p_d(a_1, \dots, a_n, x)/\mathfrak{D}(a_1, \dots, a_n, x) \text{ and } p_\ell(a_1, \dots, a_n, x)$$

consist of monomials with the same exponent vectors. In particular, they contain equally many monomials and have equal Newton polytopes.

The following table illustrates Conjecture 14. It gives the linear ordinary differential operator whose solution space is spanned by the branches of an implicitly defined algebraic function $y = y(x)$, the order of this operator and the multidegree of its leading coefficient with respect to x and the parameters of equation listed in lexicographic order.

Table 1. Computation times for annihilating operators and their properties

Algebraic curve	The annihilating operator for $y = y(x)$	Ord.	Leading coeff.	Comput. time (sec.)
$y^4 + ay^3 + x = 0$	$(27a^4x^2 - 256x^3)d_x^4 + 4x(27a^4 - 416x)d_x^3 + 60(a^4 - 36x)d_x^2 - 360d_x$	4	(2,4)	0.374
$y^4 + ay^3 + by^2 + x = 0$	$x(14b^3 - 4a^2b^2 + 8bx - 3a^2x)(16b^4 - 4a^2b^3 - 128b^2x + 144a^2bx - 27a^4x + 256x^2)d_x^4 + \dots + 120(74b^3 - 21a^2b^2 + 24bx - 9a^2x)d_x$	4	(4,6,7)	1.03

Algebraic curve	The annihilating operator for $y = y(x)$	Ord.	Leading coeff.	Comput. time
$y^4 + ay^3 + by^2 + cy + x = 0$	$(45c^2 + 14b^3 - 47abc - 4a^2b^2 + 12a^3c + 8bx - 3a^2x)(27c^4 + \dots - 256c^3)d_x^4 + \dots + 120(243c^2 + 74b^3 - 249abc - 21a^2b^2 + 63a^3c + 24bx - 9a^2x)d_x$	4	(4,7,7,6)	3.011
$y^5 + ay + x = 0$	$(256a^5 + 3125x^4)d_x^4 + 31250x^3d_x^3 + 73125x^2d_x^2 + 31875xd_x - 1155$	4	(4,5)	2.012
$y^5 + ay^2 + by + x = 0$	$(51200b^6 - 15930a^4b^3 - 2187a^8 + 68000ab^4x - 1350a^5bx - 26250a^2b^2x^2)(256b^5 - 27a^4b^2 - 1600ab^3x + 108a^5x + 2250a^2bx^2 + 3125x^4)d_x^4 + \dots + 120(492800b^6 - 139995a^4b^3 - 16038a^8 + 222000ab^4x + 8100a^5bx - 39375a^2b^2x^2)$	4	(6,13,11)	5.008
$y^5 + ay^3 + by^2 + cy + x = 0$	$(102400c^6 + \dots - 2500a^3bx^3)(256c^5 - 27b^4c^2 + 144ab^2c^3 + \dots + 3125x^4)d_x^4 + \dots - 120(985600c^6 + \dots - 625a^3bx^3)$	4	(7,15, 13,11)	31.668
$y^5 + ay^4 + x = 0$	$(265a^5x^3 + 3125x^4)d_x^5 + 5x^2(384a^5 + 6875x)d_x^4 + 780x(4a^5 + 125x)d_x^3 + 60(14a^5 + 1175x)d_x^2 + 6720d_x$	5	(4,5)	10.436
$y^5 + ay^4 + by^3 + x = 0$	$(1680ab^9 + \dots + 56a^9x^2)(108b^5x^2 - 27a^2b^4x^2 + 2250b^2x^3 - 1600a^3bx^3 + 256a^5x^3 + 3125x^4)d_x^5 + \dots + 1680(20160ab^9 + \dots + 224a^9x^2)d_x$	5	(6,15,14)	38.563
$y^5 + ay^4 + by^3 + cy^2 + x = 0$	$(160380c^8 + \dots - 56a^9x^3)(108c^5x + 16b^3c^3x + \dots + 3125x^4)d_x^5 + \dots + 1680(2779920c^8 - 1242720b^3c^6 + \dots - 224a^9x^3)d_x$	5	(7,16, 14,13)	279.242
$y^5 + ay^4 + by^3 + cy^2 + dy + x = 0$	$(4928000d^6 + \dots + 56a^9x^3)(256d^5 - 27c^4d^2 + 144bc^2d^3 + \dots + 3125x^4)d_x^5 + \dots + 1680(9011200d^6 + \dots + 224a^9x^3)d_x$	5	(7,17,15, 13,11)	3038.47
$y^6 + ay^3 + by^2 + cy + x = 0$	$(6398437500c^{10} + \dots + 67184640a^2bc^2x^5)(3125c^6 + 256b^5c^2 + \dots - 46656x^5)d_x^5 + \dots + 5040(72942187500c^{10} + \dots + 13436928a^2bc^2x^5)$	5	(10,18, 17,16)	799.427

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Многогранник Ньютона оптимального зануляющего оператора, связанного с алгебраической кривой

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В статье изучается линейный дифференциальный оператор с полиномиальными коэффициентами, чье пространство голоморфных решений порождается всеми ветвями алгебраической функции, заданной алгебраическим уравнением с коэффициентами общего положения. Основным результатом состоит в описании коэффициентов этого оператора в терминах их многогранников Ньютона.

Ключевые слова: алгебраическая функция, минимальный дифференциальный оператор, многогранник Ньютона.