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Visualizing Algebraic Curves: from Riemann to Grothendieck

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We consider the smallest possible ramification. The corresponding pairs are represented by only finite set of points in the individual Hurwitz space, but the set of Riemann surfaces admitting the meromorphic functions with the smallest possible number of critical values is dense in the moduli space.

Key words: Riemann surface, algebraic curves, Hurwitz space.

Introduction

0.0. Riemann and Riemann surfaces. Some of Riemann's ideas were about a century ahead of his time. Here is a short list of them related to the surfaces bearing Riemann's name.

• The *definition* of a Riemann surface transformed a vague concept of a multi-valued (analytic) function into the concept of a usual function defined on something *spread over the complex line.* During the XXth century this *something* was further transformed into the *definition* of a topological surface endowed with the sheaf of holomorphic functions.

• The definition of *genus* was one of the first rigorously defined topological concepts, and it turned out to be closely related to algebraic geometry and complex analysis.

• The seemingly technical problem of calculating the dimensions of spaces of meromorphic functions with a prescribed pole structure was completely solved by Riemann himself and his student Gustav Roch; it turned out to be the origin of one of the most powerful generalizations of the XXth century – the *Riemann-Roch-Hirzebruch-Grothendieck. . .* theorem.

• The idea of *classification* of abstract Riemann surfaces up to *isomorphism* inspired Riemann to introduce one of the most mysterious objects of modern math, the *moduli* spaces $\mathfrak{M}_g$ of Riemann surfaces of genus g, and to calculate their dimensions. The complete understanding of moduli spaces (e.g., their *homologies*) is still out of the scope of our understanding.

• *Riemann's existence theorem* identified these newly introduced Riemann surfaces with the well-developed theory of algebraic curves.

This list can be easily continued (abelian integrals, theta functions, . . .). The theory of Riemann surfaces is one of several from Riemann's legacy that has developed into a deep branch of mathematics and is quite active nowadays. It is, however, unique in one aspect: its objects are as visualizable as, say, the triangles in the elementary geometry (the zeros of the zeta function are not). The present paper, as its title suggests, is mainly devoted to the details of the visualization of Riemann surfaces. There is, however an important difference between the Riemann's and the Grothendieck's approach. Riemann thought about his surfaces in the context of *continuous* mathematics; therefore, the genuine Riemann surfaces are subject to continuous variation of *parameters* — usually complex numbers. As for the visualization tools introduced by Grothendieck (*dessins d'enfants,* see below), they are intended to "store" the complex structure on a surface in the finite amount of graphical information. Therefore, not all the Riemann surfaces are *storable* by dessins

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d'enfants. It turns out, that the Grothendieck's method of visualizing a Riemann surface is applicable exactly in the cases when a surface has an *arithmetical* nature; in particular, these surfaces are dense among all the Riemann surfaces.

0.1. Riemann surfaces: two meanings of the term. Sometimes people speak about the Riemann surface *of an analytic function* (or rather of a *germ of a holomorphic function*) as about the maximal domain on which its analytic continuation is defined. Sometimes the term *Riemann surface* means just the *one-dimensional complex manifold,* with no particular function specified; it is convenient to refer to this as an *abstract* Riemann surface.

In the present paper we discuss only *compact* Riemann surfaces, hence when it is a Riemann surface *of a function* the function is *algebraic.* From the modern point of view we consider a pair $(\mathcal{X}, f)$ where $\mathcal{X}$ is a compact Riemann surface and f is a nonconstant meromorphic function on it, i.e. a *holomorphic* mapping onto the Riemann sphere

$$f : \mathcal{X} \longrightarrow \mathbf{P}^1(\mathbb{C});$$

classically $\mathcal{X}$ is a Riemann surface of the inverse *multivalued* function $f^{-1\circ}$. Sometimes it is preferable to think about f as a *meromorphic* function

$$f : \mathcal{X} \setminus f^{-1\circ}(\infty) \longrightarrow \mathbb{C}.$$

For all but a finite number of points $P \in \mathbf{P}^1(\mathbb{C})$ the cardinality of the set $\#f^{-1\circ}(P)$ is the same; it is called the *degree* of f and denoted $\deg f$. The points $P \in \mathbf{P}^1(\mathbb{C})$ for which

$$\#f^{-1\circ}(P) < \deg f$$

are called the *critical values* of f; the set of all of them will be denoted $\mathrm{CritVal}(f)$. Alternatively the critical values of f are called the *ramification points* of f, thought of as a covering of the Riemann's sphere.

The definition of *isomorphism* between abstract Riemann surfaces is obvious; the isomorphism classes of abstract Riemann surfaces of genus g constitute the above-mentioned *moduli space* $\mathfrak{M}_g$. However, for the definition of the isomorphism between $(\mathcal{X}, f)$ and $(\mathcal{X}', f')$ there are *two* natural possibilities: either $(\mathcal{X}, f)$ and $(\mathcal{X}', f')$ are called isomorphic whenever there exists an isomorphism

$$\phi : \mathcal{X} \longrightarrow \mathcal{X}'$$

such that the diagram

$$\mathcal{X} \xrightarrow{\phi} \mathcal{X}'$$
$$f \searrow \qquad \swarrow f'$$
$$\mathbf{P}^1(\mathbb{C})$$

commutes, or they are called isomorphic if there exist such an isomorphism

$$\phi : \mathcal{X} \longrightarrow \mathcal{X}'$$

and such a fractional-linear transformation T of the Riemann sphere that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & \mathcal{X}' \\ \downarrow f & & \downarrow f' \\ \mathbf{P}^1(\mathbb{C}) & \xrightarrow{T} & \mathbf{P}^1(\mathbb{C}) \end{array}$$

commutes.

Denoting $d = \deg f$, call the *big Hurwitz space* $\mathfrak{HUR}_{g,d}$ the set of classes of isomorphism in the sense of the triangular diagram and *small Hurwitz space* $\mathfrak{Hur}_{g,d}$ the set of classes of isomorphism in the sense of the square diagram.

0.2. Riemann's count. To illustrate Riemann's style of thinking we reproduce briefly[1] the in-

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[1]Simplifying a bit the exposition in [4]

formal calculation of $\dim \mathfrak{M}_g$, the dimension of moduli spaces of compact Riemann surfaces of genus $g > 1$. According to Riemann-Roch, any such surface $\mathcal{X}$ admits a non-constant meromorphic function f with the *simple* poles in the $g + 1$ *generic* points; the pairs

$$(\mathcal{X}, f) \in \mathfrak{HUR}_{g,g+1}$$

depend on

$$\dim \mathfrak{M}_g + g + 1 + 2$$

parameters (the last 2 for choosing f among $af + b$'s). Denote the number of critical values of f by

$$r := \#\mathrm{CritVal}(f)$$

(according to our assumption $\infty \in \mathbf{P}^1(\mathbb{C})$) is *not* a critical value). Now we suppose that $(\mathcal{X}, f)$ is *generic* in the sense that all the r critical values of f are different and *simple* – i.e., their preimages consist of $\deg f - 1$ points, or, equivalently, the differential $\mathrm{d}f$ has a simple zero in each of them; we have

$$2g - 2 = \deg(\mathrm{d}f) = r - 2(g + 1),$$

so

$$r = 4g.$$

Now the pair $(\mathcal{X}, f)$ is restorable by the r critical points up to a common *affine* transformation and some *discrete* information (monodromy). So the (intuitively understood) space of such classes of pairs [2] has the dimension

$$\dim \mathfrak{M}_g + g + 1 = 4g - 2,$$

which gives the correct answer

$$\dim \mathfrak{M}_g = 3g - 3.$$

0.3. Drawing elements of Hurwitz spaces. The pairs $(\mathcal{X}, f)$ are better suited for the visualization than the abstract Riemann surfaces $\mathcal{X}$.

It was Riemann who suggested in [10] imagining a pair $(\mathcal{X}, f)$, where f is a covering of degree d ramifying over r points as d copies of the Riemann sphere pasted along some cuts that connect (identically on all the copies) the ramification points. What is difficult nowadays is to define these cuts and pasting rigorously (understanding that the often-mentioned *sheets* of Riemann surfaces are not well defined) and to supply the results of pasting by the appropriate structures.

The modern version of the Riemann's picture is more formal but comparably clear. We choose the point $\star \in \mathbf{P}^1(\mathbb{C}) \setminus \#\mathrm{CritVal}(f)$ and consider the *monodromy representation*

$$\pi_1(\mathbf{P}^1(\mathbb{C}) \setminus \{r \text{ points}\}, \star) \longrightarrow \mathrm{Aut}[f^{-1\circ}\star],$$

that abstractly is a conjugate class of the group morphisms

$$\mathfrak{Free}_{r-1} \longrightarrow \mathfrak{S}_d,$$

where $\mathfrak{Free}_s$ denotes the free group with s generators and $\mathfrak{S}_d$ is a permutation group on d letters. Equivalently, the monodromy representation is defined by specializing $r - 1$ permutations of d elements (classically *sheets* of $\mathcal{X}$ over $\star$); the condition of *connectedness* of $\mathcal{X}$ is equivalent to the *transitiveness* of the group generated by these permutations.

So the points of $\mathcal{X}$ that are not the critical points of f can be thought of as just certain homotopical classes of the paths on the punctured Riemann sphere. In the remaining part of this paper we call the *drawing* any way of graphical representing of the complex structure on a surface.

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[2]It is *between* our big and small Hurwitz spaces

0.4. Stratification of Hurwitz spaces by ramification. Denote

$$\mathfrak{HUR}_{g,d;r} := \{(\mathcal{X}, f) \in \mathfrak{HUR}_{g,d} \mid \#\mathrm{CritVal}(f) = r\}$$

and

$$\mathfrak{Hur}_{g,d;r} := \{(\mathcal{X}, \underline{f}) \in \mathfrak{Hur}_{g,d} \mid \#\mathrm{CritVal}(f) = r\},$$

where $\underline{f}$ is the equivalence class of fractional-linear transformations of f. Then

$$\mathfrak{HUR}_{g,d} =: \coprod_{r=3}^{2g+2d-2} \mathfrak{HUR}_{g,d;r},$$

and

$$\mathfrak{Hur}_{g,d} =: \coprod_{r=3}^{2g+2d-2} \mathfrak{Hur}_{g,d;r},$$

the upper number found from $2g - 2 = \deg(\mathrm{d}f) = r - 2d$ for a *generic* f.

0.5. The subject of the paper. We are going to consider the *smallest* possible ramification. The corresponding pairs are represented by only finite set of points in the individual Hurwitz space, but the set of Riemann surfaces admitting the meromorphic functions with the smallest possible number of critical values is dense in the moduli space. In other words, the *union* of such pairs is sent by the *forgetting* map

$$\coprod_{d=2}^{\infty} \mathfrak{HUR}_{g,d} \longrightarrow \mathfrak{M}_g : (\mathcal{X}, f) \mapsto \mathcal{X}$$

to the dense set. These are the curves we are going to visualize.

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Morally this paper is the continuation of [11] and is dedicated to the memory of my father, the great admirer of Riemann and his style of thinking.

1. Belyi Pairs and Dessins d'Enfants

1.0. Moduli spaces over arbitrary fields. It follows from Riemann's existence theorem that most of the above can be reformulated by substituting "compact Riemann surface" by "smooth projective complex algebraic curve" and "meromorphic" by "rational". After that we can generalize most of the above discussion (but *drawing*) to the case of arbitrary field $\mathbb{K}$, which is assumed to be algebraically closed. So $\mathfrak{M}_g$ is substituted by $\mathfrak{M}_g(\mathbb{C})$ and $\mathfrak{M}_g(\mathbb{K})$ acquires meaning for an arbitrary algebraically closed field $\mathbb{K}$.

1.1. The smallest possible ramification. The cases $r = 1$ and $r = 2$ are trivial, so we concentrate on the case $r = 3$. In this case we introduce some special VISUALIZATION tools – basically following [5] but adding some technical details.

1.2. Belyi sphere. Color the Riemann sphere $\mathbf{P^1}(\mathbb{C})$ in the following way.

The upper half-plane will be *white* and the lower half-plane *black*. The (projective) real line will be considered as a topological triangle on the Riemann sphere with the vertices 0,1,∞. The negative numbers will be *blue* (associated with *cold*), the interval (0.1) *green* and the numbers greater than one *red* (associated with *hot*). The vertices are of the same color as the opposite sides.

By the *Belyi sphere* we understand the Riemann sphere colored as described.

1.3. Colored dessins d'enfants. The Belyi sphere just introduced is an example of *colored dessin d'enfant.* By definition, such an object is a triangulation of a compact surface, where the triangles are black and white, and it is supposed that
- all the adjacent triangles are colored differently (it is equivalent to the *orientability* of the surface);
  - every triangle contains blue, green and red vertices;
  - every triangle contains blue, green and red sides;
  - the incident objects are colored differently.

1.4. Drawing Belyi pairs. A non-constant meromorphic function on a Riemann surface is called a *Belyi function* if it has no more than three critical values. A *Belyi pair* is a Riemann surface together with a Belyi function on it.

Given a Belyi pair $(\mathcal{X}, \beta)$ over $\mathbb{C}$ we assign to each point $P \in \mathcal{X}$ the color of $\beta(P)$. It is easy to check that every complex Belyi pair thus produces a colored dessin d'enfant.

Conversely, any colored dessin d'enfant can be thought of as a scheme of a covering of the Belyi sphere — any triangle is supposed to be mapped homeomorphically onto the half of the sphere corresponding to its color, and these homeomorphisms are pasted by the obvious regulations defined by colors. On a surface carrying a colored dessin d'enfant there is a unique complex structure with respect to which these homeomorphisms are *conformal;* hence any colored dessin [3] represents some compact Riemann surface.

Due to the typographical restrictions we are unable to use colors in our examples. The black-and-white version of drawing *some* of Belyi pairs is explained below.

1.5. Dessins and curves over number fields. Arithmetic comes in unexpectedly.

Theorem 1 ([3], [5])**.** *The above association of the colored dessin to a complex Belyi pair establishes the bijection*

$$\{colored\ dessins\} \leftrightarrow \coprod_{g=0}^{\infty} \coprod_{d=1}^{\infty} \mathfrak{Hur}_{g,d;3}(\overline{\mathbb{Q}}).$$

(To make the statement precise, we should identify the 1-point sets $\mathfrak{Hur}_{0,1}$ and $\mathfrak{Hur}_{0,1;3}$). More accurate formulation in terms of equivalence of suitable categories can be found in [12].

There exists a non-colored version of this theorem as well.

A Belyi pair is called *pure* if all its ramifications over 1's are of order 2.

All the blue vertices of the colored dessin corresponding to a pure Belyi pair have valencies 2 and carry no information (just lying in the "midpoints" of green edges). All the corresponding colored dessin can be restored by green edges and red vertices. This is what we are going to draw in black in our non-colored dessins.

Non-colored dessins are called simply *dessins;* a dessin can be alternatively defined as a graph, embedded into a compact connected oriented closed surface, such that the complement to the image of the graph is homeomorphic to the disjoint union of discs.

The black-and-white analog of the above colored theorem establishes the equivalence between dessins and pure Belyi pairs, graphs being the preimages of the segment [0,1] on the Riemann sphere. The details can be found in many texts, see, e.g., [8] and the references therein. It should be noted, however, that many authors understand by dessins d'enfants the objects that are intermediate between our colored and black-and-white dessins. In our language they consider the

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[3]From now on we omit *enfants*

colored dessins but "see" only green edges and the adjacent (blue and red, that they call *black* and *white*) vertices.

Restriction by only pure dessins is not a severe one since for any Belyi function β the function $4\beta(1-\beta)$ is a pure Belyi function.

2. Visualizing Riemann Surfaces by Dessins d'Enfants

2.0. Some simple examples. The generalized Fermat curves defined by the affine equations

$$x^m + y^n = 1$$

provide us with a 2-parametric family of examples since

$$\beta := x^m = 1 - y^n$$

are (non-pure) Belyi functions.

The Klein quartics K, defined by the projective equation

$$x^3 y + y^3 z + z^3 x = 0$$

gives a beautiful example since

$$K \longrightarrow K/\mathrm{Aut}K$$

is pure Belyi. Actually it was drawn by Felix Klein in [6].

2.1. Belyi height. Since on any algebraic curve over the field of algebraic numbers there is at least one Belyi function, we introduce the *smallest* degree of such a function as a measure of *complexity* of curves. Namely, we define the *Belyi height* as the function

$$h_{Bel} : \coprod_{g \in \mathbb{N}} \mathfrak{M}_g(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N}$$

by

$$h_{Bel}(\mathcal{X}) := \min\{d \mid \exists \ pure \ \beta : \mathcal{X} \longrightarrow \overline{\mathbb{Q}} \text{ such that } (\mathcal{X}, \beta) \in \mathfrak{HUR}_{g,d;3}.\}$$

The Riemann surfaces, corresponding to the curves with bounded Belyi height, can be identified by the limited amount of graphical information.

Recently all the curves $\mathcal{X}$ of the Belyi height

$$h_{Bel}(\mathcal{X}) \leq 4$$

have been listed and the corresponding Belyi functions calculated, see [1]. Besides $\mathbf{P}^1$, they consist of 52 curves of genus 1 and 4 curves of genus 2.

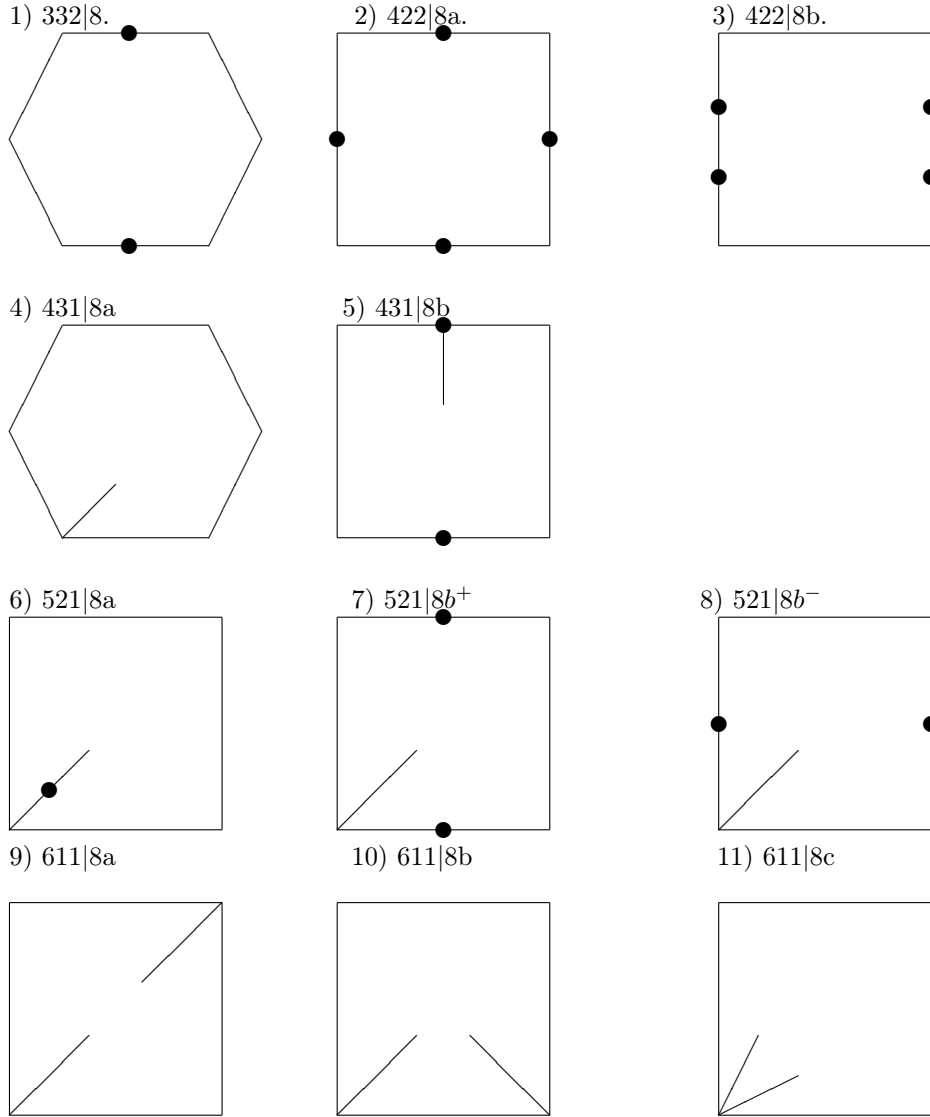The part of this calculation is reproduced in the next section.

2.2. Unicellular toric dessins. We consider connected graphs with 4 edges embedded into the tori; the word *unicellular* means that the complement is homeomorphic to a disc. Most of the calculations are new, some very hard.

The labels of dessins in the table below have the form $a_1 a_2 a_3 | 8$ where a_1, a_2, a_3 are the 0-valencies and 8 is the common 2-valency.[4] When the lists of valencies have several realizations, the notations like $a_1 a_2 a_3 | 8n$ or $a_1 a_2 a_3 | 8n^{\pm}$ are used; $a_1 a_2 a_3 | 8n^{+}$ and $a_1 a_2 a_3 | 8n^{-}$ correspond to mirror symmetric dessins.

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[4]By a 0-*valency* of a vertice we mean the number of *germs* of edges incident to it (the loop incident to a vertice is counted twice and the other edges counted once). The 2-*valency* of a face of a dessin generally means the number of edges lying in the closure of a face but in the present context all the 2-valencies equal 8.

All the toric dessins are drawn either inside the square or inside the hexagon; it is assumed that in both cases the opposite sides are identified. All the vertices of squares and hexagons are assumed to be the vertices of the graphs; the remaining vertices (usually of valency 2) are reproduced by the bullets.

Dessins

1) 332|8.

2) 422|8a.

3) 422|8b.

4) 431|8a

5) 431|8b

6) 521|8a

7) 521|8b$^+$

8) 521|8b$^-$

9) 611|8a

10) 611|8b

11) 611|8c

In the table below we collect some arithmetical information on all the eleven above dessins. By a *bad prime* we mean simply a divisor of the denominator norm of the j−invariant[5] of the corresponding elliptic curves.

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[5]Recall that the j−invariant of an elliptic curve defined by an equation in the Legendre form $y^2 = x(x-1)(x-t)$ is $j = 256 \frac{(t^2-t+1)^3}{t^2(t-1)^2}$

Number	Dessin	j	Bad primes
1	$332\vert 8$	$\frac{219488}{729} = -\frac{2^5 19^3}{3^6}$	3
2	$422\vert 8a$	$1728 = 2^6 3^3$	none
3	$422\vert 8b$	$10976 = 2^5 7^3$	none
4	$431\vert 8a$	$\frac{207646}{6561} = \frac{2 \cdot 47^3}{3^8}$	3
5	$431\vert 8b$	$\frac{2^5 5^3 13859^3}{3^8 7^{10}}$	3,7
6	$521\vert 8a$	The real root of I_{521}	2,5,7
7	$521\vert 8b^+$	One of the non-real roots of I_{521}	2,5,7
8	$521\vert 8b^-$	One of the non-real roots of I_{521}	2,5,7
9	$611\vert 8a$	$\frac{4000}{9} = \frac{2^5 5^3}{3^2}$	3
10	$611\vert 8b$	One of the roots of I_{611}	2,3,7
11	$611\vert 8c$	One of the roots of I_{611}	2,3,7

Here

$$I_{521} = 2^{15} 5^{14} 7^{10} j^3 - 31562956092228535000000000000 j^2 +$$

$$+ 7482958853213479960732972656253 j - 564055135320668135938721399828128$$

and

$$I_{611} = 2^5 3^4 7^{10} j^2 - 2093223975378624 j + 1397111322473062559.$$

We have considered several Riemann surfaces that are selected by Nature as the *simplest* ones in the sense that they can be drawn just by several lines on the torus. As we see, some of them are *not at all* simple from the point of view of the formulas![6]. In more scientific terms, a low Belyi height can coexist with terrible classical (say, Weil or Neron-Tate) height. The relations between various heights of the curves over number fields — i.e., basically, the problem of the *amount of information* needed to define a curve — is, hopefully, the subject of future investigations; for the time being the author is aware only of some crazy estimates of ones in terms of the others.

The above does not mean that our eleven curves are *arbitrary*. On the contrary, they obey some evident regulations. E.g., the primes of bad reduction demonstrate the obvious *combinatorial* sense. It may seem not to be true concerning the prime 7 that is present almost everywhere and does not divide any obvious valencies; however, in the simpler case of plane trees the similar phenomenon was explained by Vashevnik in [16]. Briefly, in our case 7=4+3=5+2=6+1.

The complete understanding even of this simple case obviously needs the blending of complex analysis and arithmetic. *Arakelov geometry* (see, e.g., [15]), hopefully, provides an appropriate frame for it.

3. Transcendent Ways of Drawing Riemann Surfaces

3.0. Brief overview. As we have mentioned, Klein started the whole story in [6] in 1870, explaining the relation of the quadric bearing his name to the tesselation of the upper half-plane by non-euclidean triangles with the angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{7}$. He asked Poincaré whether this example admits generalizations; the answer YES!!! was the beginning of the general uniformization theory. Lots of texts concerning the uniformization of dessins d'enfants — i.e., drawing a dessin on its universal covering — appeared during the last years; see, e.g., [14] and the references therein.

See [13] for defining Riemann surfaces in terms of piecewise-euclidean metrics, especially equilateral triangulations.

Several other approaches exist — through jacobians, algebraic solutions of Painlevé-6, etc.

3.1. Decomposition into semi-infinite cylinders. This approach is especially important for

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[6]We do not give the equations of these curves, referring to [1]; however, the reader can imagine the length of the equations that result in some of the above $j-$invariants

understanding the geometry of the moduli space. See [7] for the basic construction and [9] for understanding the position of arithmetical curves.

3.2. Decomposition into finite cylinders. Here is, finally, the recent original result: an explicit family of Strebel differentials was constructed in [2].

Theorem 2. *Let $g \in \{2, 4, 6, \dots\}$. For each $p \in \mathbb{R}$ denote*

$$\ddot{X}_p : y^2 = x^{2g+2} - 2px^{g+1} - 1$$

and X_p its smooth complete model. Then the quadratic differential $x^{g-1} \left(\dfrac{dx}{y} \right)^2$ is regular Strebel on X_p.
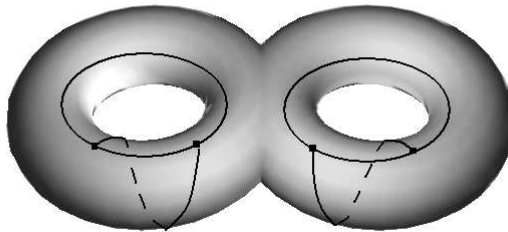
This picture shows the structure of separatrices in the case $g = 2$. It is *not* a dessin d'enfant: it cuts the surface into the straight cylinder (on the metric defined by the quadratic differential) with the hexagonal boundaries. The surface is restored by pasting the opposite sides of each of the hexagons.

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