УДК 517.955, 517.55

Boundary Problems for Helmholtz Equation and the Cauchy Problem for Dirac Operators

Alexander A. Shlapunov*
Institute of Mathematics,
Siberian Federal University,
Svobodny, 79, Krasnoyarsk, 660041
Russia

Received 01.12.2010, received in revised form 01.12.2010, accepted 15.01.2011

Studying an operator equation $Au = f$ in Hilbert spaces one usually needs the adjoint operator $A^\star$ for A. Solving the ill-posed Cauchy problem for Dirac type systems in the Lebesgue spaces by an iteration method we propose to construct the corresponding adjoint operator with the use of normally solvable mixed problem for Helmholtz Equation. This leads to the description of necessary and sufficient solvability conditions for the Cauchy Problem and formulae for its exact and approximate solutions.

Keywords: mixed problems, Helmholtz equation, Dirac operators, ill-posed Cauchy problem.

Many problems in applications of mathematics can be formulated on the language of Functional Analysis as the study of operator equations of the first type in Hilbert spaces (cf. [1,2]). Namely, let Hilbert spaces H_1 and H_2 and a (continuous linear) map $A : H_1 \to H_2$ be fixed. Then the problem is the following: given $f \in H_2$ find $u \in H_1$ with $Au = f$. Though the problem is usually ill-posed (see [1,2]), there are many approaches to it, for instance, fixed points method, iteration methods, methods of bases with double orthogonality property, method of small parameter perturbation, etc. Regarding the Cauchy problem for elliptic systems we refer, for instance, to [3–7]. Usually one needs the adjoint $A^\star$ for A in the sense of Hilbert spaces in order to write down proper solvability conditions. Then there are many possibilities to construct regularization of the problem (i.e. a family of approximate solutions depending on a parameter and converging to an exact solution if and only if a solution exists).

Dirac type operators naturally appear in many applications (see, for instance, [8,9]). These include Cauchy-Riemann operator, gradient operator, operator Moisil-Teodorescu, stationary Maxwell operator and so on. We propose to consider the ill-posed Cauchy problem for Dirac type operators, see [7,22,23] in Lebesgue spaces. The Dirichlet problem for Helmholtz Equation plays an essential role in the formulation of the Cauchy problem as an operator equation of the first type in Hilbert spaces. Then we construct the corresponding adjoint operator with the use of a normally solvable mixed problem for Helmholtz Equation (cf. [6] for elliptic operators in Sobolev spaces where the Dirichlet Crack Problem for the Laplace equation was used instead).

1. Preliminaries

Let $\mathbb{R}^n$ be n-dimensional Eucledian space and $\mathbb{C}^n$ be n-dimensional complex space with points being n-vectors $z = (z_1, ..., z_n)$, where $z_j = x_j + \sqrt{-1}x_{j+n}$, $j = 1, ..., n$, $x = (x_1, ..., x_{2n}) \in \mathbb{R}^{2n}$ and $\sqrt{-1}$ being imaginary unit. Let $A = \sum\limits_{j=1}^{n} a_j \dfrac{\partial}{\partial x_j}$ be a Dirac operator in $\mathbb{R}^n$, i.e., such a

*ashlapunov@sfu-kras.ru

homogeneous first order matrix differential operator with constant coefficients that

$$\sum_{j=1}^{n}\sum_{m=1}^{n} a_j^* a_m \xi_j \xi_m = |\xi|^2 I_k \text{ for all } \xi \in \mathbb{R}^n; \tag{1}$$

here a_j are $(l \times k)$-matrices of complex numbers, $a_j^* = \overline{a_j^T}$ are their adjoint matrices and I_k is the identity $(k \times k)$-matrix. In particular, the symbol $\sigma(A)(\xi) = \sum_{j=1}^{n} a_j \xi_j$ is injective as the map from $\mathbb{C}^k$ to $\mathbb{C}^l$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, i.e. $l \geqslant k$. We say that A is elliptic if $l = k$ and overdetermined elliptic if $l > k$. Denote A^* and A^T the formally adjoint and transposed operators for A respectively: $A^* = -\sum_{j=1}^{n} a_j^* \dfrac{\partial}{\partial x_j}$ and $A^T = -\sum_{j=1}^{n} a_j^T \dfrac{\partial}{\partial x_j}$. Then $A^*A = -\Delta I_k$ where $\Delta = \sum_{j=0}^{n} \dfrac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^n$.

Typical elliptic Dirac operators are the derivative operator $\dfrac{d}{dx}$ in $\mathbb{R}$ and the (doubled) Cauchy-Riemann operator $2\dfrac{\partial}{\partial \overline{z}} = \dfrac{\partial}{\partial x_1} + \sqrt{-1}\dfrac{\partial}{\partial x_2}$ in $\mathbb{C}$. Typical overdetermined Dirac operators are the gradient operator $\nabla = -\left(\dfrac{\partial}{\partial x_1}, \dots \dfrac{\partial}{\partial x_n}\right)^T$ in $\mathbb{R}^n$, the (doubled) multi-dimensional Cauchy-Riemann system $2\overline{\partial} = -2\left(\dfrac{\partial}{\partial \overline{z}_1}, \dots \dfrac{\partial}{\partial \overline{z}_n}\right)^T$ in $\mathbb{C}^n$ and the stationary Maxwell system $M = -(\mathrm{rot}, \mathrm{div})^T$ in $\mathbb{R}^3$. For Dirac operators on manifolds and in the Quantum Physics see, for instance, [8,9]. Our approach is also fit on a manifold but this lead to a mixed problem for more complicated second order elliptic operator.

Let D be a bounded domain (i.e. open connected set) in $\mathbb{R}^n$, and let $\overline{D}$ be its closure. We always assume that the boundary ∂D of D is of class C^∞. As usual we denote by $\mathcal{D}(D)$ the space of all the smooth functions with compact supports in D and by $\mathcal{D}'(D)$ the space of distributions over D. Besides, let $C^\infty(\overline{D})$ stand for the set of smooth functions in D with any derivative extending continuously to $\overline{D}$ and $C^\infty_\sigma(\overline{D})$ stand for the set of functions vanishing on a closed subset $\sigma \subset \overline{D}$.

Let $E = \mathbb{R}^n \times \mathbb{C}^k$ be the trivial k-vector bundle. The set of all k-vector functions over domain D with the components from a functional space $\mathfrak{S}(D)$ will be denoted by $\mathfrak{S}(D, E)$.

It is known (see, for instance, [12]) that A induces a differential compatibility complex:

$$0 \longrightarrow C^\infty(E_0) \xrightarrow{A_0} C^\infty(E_1) \xrightarrow{A_1} C^\infty(E_2) \xrightarrow{A_2} \dots \xrightarrow{A_{N-1}} C^\infty(E_N) \longrightarrow 0 \tag{2}$$

where A_i are differential operators with constsant coefficients, $A_0 = A$ and $A_{i+1} \circ A_i \equiv 0$; here $E_i = \mathbb{R}^n \times \mathbb{C}^{k_i}$ are trivial vector bundles of ranks k_i (of course, $k_0 = k$, $k_1 = l$). From now on we assume that complex (2) is elliptic and orders of all the operators equal to one, i.e. $A_i = \sum_{j=1}^{n} a_j^{(i)} \dfrac{\partial}{\partial x_j} + a_0^{(i)}$ where $a_j^{(i)}$ are $(k_{i+1} \times k_i)$-matrices of complex numbers. It is true in many cases, though it is not known to be true in general.

For instance, if $A = \nabla$ then the corresponding sequence is de Rham complex with $E_i = \Lambda^i$ $(0 \leqslant i \leqslant n)$ being the set of all the exterior differential forms of degree i in $\mathbb{R}^n$ and $A_i = d_i$ being exterior derivatives for the forms. Similarly, if $A = 2\overline{\partial}$ then the corresponding sequence is Dolbeault complex with $E_i = \Lambda^{(0,i)}$ $(0 \leqslant i \leqslant n)$ being the set of all the complex exterior forms of bi-degree $(0, i)$ in $\mathbb{C}^n$ and $A_i = \overline{\partial}_i$ be the (graduated) Cauchy-Riemann operator extended to the differential forms. Obviously, if $l = k$ then $E_0 \cong E_1$ and $A_i = 0$ for all $i \geqslant 1$.

We write $L^2(D)$ for the Hilbert space of all the measurable functions in D with a finite norm $(u,v)_{L^2(D)} = \int_D u(x)\overline{v}(x)\,dx$. Then the Hermitian form

$$(u,v)_{L^2(D,E_i)} = \int_D v^*(x)u(x)\,dx$$

defines the Hilbert structure on $L^2(D,E_i)$. We also denote $H^s(D)$ the Sobolev space of all the distributions over D, whose weak derivatives up to the order $s \in \mathbb{N}$ belong to $L^2(D)$. The closures of $\mathcal{D}(D)$ and $C^\infty_{\partial D}(\overline{D})$ in $H^s(D)$ will be denoted by $H^s_0(D)$ and $H^s_{\partial D}(D)$ respectively; for $s=1$ these spaces coincide.

For non-integer positive $s \in \mathbb{R}_+$ we define Sobolev spaces H^s with the use of the standard interpolation procedure (see [14] or [5, §1.4.11]). It is known (see, for instance, [14]) that functions of $H^s(D)$, $s \in \mathbb{N}$, have traces on ∂D of class $H^{s-1/2}(\partial D)$ and the corresponding trace operator is continuous. We will use Sobolev spaces of fractional smoothness for boundary data only.

Sobolev spaces of negative smoothness may be defined in many different ways (see [15], [16]). We follow [16] and consider Sobolev spaces $H^{-s}(D)$ and $H(D, |\cdot|_{-s})$, $s \in \mathbb{R}_+$, being the completions of $C^\infty(\overline{D})$ with respect to the norms

$$\|u\|_{H^{-s}(D)} = \sup_{\phi \in C^\infty(\overline{D})} \frac{|(u,\phi)_{L^2(D)}|}{\|\phi\|_{H^s(D)}}, \quad |u|_{-s,D} = \sup_{\phi \in C^\infty_{\partial D}(\overline{D})} \frac{|(u,\phi)_{L^2(D)}|}{\|\phi\|_{H^s(D)}}$$

Easily, $H^{-s}(D,E) \subset H(D, |\cdot|_{-s}, E)$. These spaces are the strong duals for $H^s(D,E)$ and $H^s_{\partial D}(D,E)$ correspondingly with respect to $L^2(D,E)$-pairing $\langle \cdot, \cdot \rangle_D$. Namely,

$$\langle u, \phi \rangle_D = \lim_{\nu \to \infty} (u_\nu, \phi)_{L^2(D,E)},\ u \in H^{-s}(D,E),\ \phi \in H^s(D,E),$$

where $\{u_\nu\} \subset C^\infty(\overline{D},E)$ is a sequence approximating u in $H^{-s}(D,E)$ (see [15] or [5, Theorem 1.4.28]). Easily, the pairing does not depend on the choice of the approximating sequence $\{u_\nu\}$ and

$$|\langle u, \phi \rangle_D| \leqslant \|u\|_{H^{-s}(D,E)} \|\phi\|_{H^s(D,E)} \text{ for all } u \in H^{-s}(D,E), \phi \in H^s(D,E). \tag{3}$$

Of course, we can do the similar procedure for the pair $H(D, |\cdot|_{-s}, E)$ and $H^s_{\partial D}(D,E)$. Clearly, as $\partial(\partial D) = 0$ then we have $H(\partial D, |\cdot|_{-s}, E) = H^{-s}(\partial D, E)$.

For a generalized vector function $u \in \mathcal{D}'(D,E)$ we always consider Au in the sense of distributions. Thus, $Au \in \mathcal{D}'(D,E_1)$ is a vector-distribution over D. However there are no reasons for it being an element of $L^2(D,E_1)$ if $u \in L^2(D,E)$. We denote by $H_A(D)$ the so called strong extension of the differential operator A, i.e. the closure of $C^\infty(\overline{D},E)$ with respect to the graph norm

$$\|u\|_{A,D} = (\|u\|^2_{L^2(D,E)} + \|Au\|^2_{L^2(D,E_1)})^{1/2}.$$

According to [13] this space coincides with the weak extension of the differential operator A, i.e. with the set of vector functions from $L^2(D,E)$ with $Au \in L^2(D,E_1)$. Of course, it is a Hilbert space with the scalar product

$$(u,v)_{A,D} = (u,v)_{L^2(D,E)} + (Au,Av)_{L^2(D,E_1)}.$$

Thus, the differential operator A induces a bounded linear operator $A: H_A(D) \to L^2(D,E_1)$ with $\|A\| \leqslant 1$. Clearly, $\|u\|_{A,D} \leqslant C_A \|u\|_{H^1(D,E)}$ for all $u \in H^1(D,E)$ with a positive constant C_A being independent on u. It is worth to mention that, due to ellipticity of A and Garding inequality (see, for instance, [16]), the closure of $\mathcal{D}(D,E)$ in $H_A(D)$ coincides with $H^1_0(D,E)$. Of course, $H_\nabla(D) = H^1(D)$, however, $H_{2\overline{\partial}}(D) \neq H^1(D)$ since the space $H_{2\overline{\partial}}(D)$ contains all the holomorphic $L^2(D)$-functions. In any case, $H_A(D) \subset H^1_{loc}(D,E)$ because of the ellipticity of A.

As $A_{i+1} \circ A_i \equiv 0$ we easily see that the differential operator A_i induces a bounded linear operator $A_i : H_{A_i}(D) \to H_{A_{i+1}}(D)$. Moreover, we easily obtain a complex of such operators

$$0 \longrightarrow H_{A_0}(D, E_0) \xrightarrow{A_0} H_{A_1}(D, E_1) \xrightarrow{A_1} H_{A_2}(D, E_2) \xrightarrow{A_2} ... \xrightarrow{A_{N-1}} L^2(D, E_N) \longrightarrow 0.$$

However it is not true that $H_{A_i}(D) \subset H^1_{loc}(D, E_i)$ for $i \geqslant 1$.

2. Traces on the Boundary

In order to formulate properly the Cauchy problem for A we need first to clarify what is the space of traces on ∂D for elements of $H_A(D)$. With this aim we denote by ρ the defining function of the domain D, i.e. $\rho \in C^\infty$, $|\nabla \rho| \neq 0$ on ∂D and $D = \{x \in \mathbb{R}^n : \rho(x) < 0\}$. Then the following Green formula holds true for $u \in C^\infty(\overline{D}, E)$, $g \in C^\infty(\overline{D}, E_1)$:

$$(A_i u, g)_{L^2(D,E_{i+1})} - (u, A_i^* g)_{L^2(D,E_i)} = \sum_{j=1}^n \int_{\partial D} g^*(x) \frac{a_j^{(i)}}{|\nabla \rho|} \frac{\partial \rho}{\partial x_j}(x)\, u(x)\, ds(x). \tag{4}$$

Set

$$\tilde{\tau}_i(u) = \sum_{j=1}^n \frac{a_j^{(i)}}{|\nabla \rho|} \frac{\partial \rho}{\partial x_j} u, \quad \tilde{\nu}_{i-1}(g) = \sum_{j=1}^n \frac{(a_j^{(i-1)})^*}{|\nabla \rho|} \frac{\partial \rho}{\partial x_j} g, \quad i \geqslant 1,$$

$\tau_i = \tilde{\nu}_{i-1} \circ \tilde{\tau}_i$. Then $\tau_0 = I_k$.

Lemma 1. *For every $u \in H_A(D)$ there is a weak trace $\tau_0(u) \in H^{-1/2}(\partial D, E)$, i.e.*

$$(Au, \psi)_{L^2(D,E_1)} - (u, A^*\psi)_{L^2(D,E)} = \langle \tau_0(u), \tilde{\nu}_0(\psi)\rangle_{\partial D} \text{ for all } \psi \in C^\infty(\overline{D}, E_1).$$

Proof. Follows from [7, Lemma 1.1]. □

According to [7, Lemma 1.2] the subspace of elements in $H_A(D)$ having zero traces coincides with $H^1_0(D, E)$. Therefore in [7] the space of traces on ∂D for elements from $H_A(D)$ was defined as the factor space $\frac{H_A(D)}{H^1_0(D, E)}$. Of course, Lemma 1 allows to identify it within $H^{-1/2}(\partial D, E)$ but we will do it more precisely. Indeed, as we have seen above, if $A = \nabla$ then $H_A(D) = H^1(D)$, i.e. in this case $\tau_0(u) \in H^{1/2}(\partial D)$. This observation, combined with Lemma 1, leads us to the following definition. For $v \in C^\infty(\partial D, E)$, set

$$\|v\|_{A,\partial D} = \sup_{C^\infty(\overline{D},E_1) \ni \psi \neq 0} \frac{\left|(v, \tilde{\nu}_0(\psi))_{L^2(\partial D,E)}\right|}{\|\psi\|_{A^*,D}}.$$

Lemma 2. *The functional $\|\cdot\|_{A,\partial D}$ defines a norm on $C^\infty(\partial D, E)$. Moreover, there are positive constants c_1, c_2 such that*

$$c_1 \|v\|_{H^{-1/2}(\partial D,E)} \leqslant \|v\|_{A,\partial D} \leqslant c_2 \|v\|_{H^{1/2}(\partial D,E)} \text{ for all } v \in C^\infty(\partial D, E).$$

In particular, the norms $\|\cdot\|_{\nabla,\partial D}$ and $\|\cdot\|_{H^{1/2}(\partial D)}$ are equivalent.

Proof. By the definition, the functional $\|\cdot\|_{A,\partial D}$ is non-negative. It is easy to see that

$$\|v_1 + v_2\|_{A,\partial D} \leqslant \|v_1\|_{A,\partial D} + \|v_2\|_{A,\partial D} \text{ for all } v_1, v_2 \in C^\infty(D, E),$$

$$\|\alpha v\|_{A,\partial D} = |\alpha| \|v\|_{A,\partial D} \text{ for all } v \in C^\infty(D, E),\ \alpha \in \mathbb{C}.$$

In particular, $\|0\|_{A,\partial D}=0$. Let now $\|v\|_{A,\partial D}=0$. Then

$$(v,\tilde\nu_0(\psi))_{L^2(\partial D,E)}=0 \text{ for all } \psi\in C^\infty(\overline{D},E_1). \tag{5}$$

Now, given u_0, h, consider Dirichlet problem for the Helmholtz operator in $D\subset\mathbb{R}^n$:

$$\begin{cases} a^2u-\Delta u &= h \quad \text{in} \quad D,\\ \qquad\quad u &= u_0 \quad \text{on} \quad \partial D,\end{cases} \tag{6}$$

where $a\in\mathbb{R}$. The problem (6) is uniquely solvable on the Sobolev scale $H^s(D)$, $s\in\mathbb{Z}_+$, and its solution is given by the Poisson type formula

$$u=\mathcal{P}_{D,a}u_0+\mathcal{G}_{D,a}h$$

for data $u\in H^{s-1/2}(\partial D)$ and $f\in H^{s-2}(D)$ $(s\geqslant 2)$ or $f\in H(D,|\cdot|_{s-2})$ $(s\leqslant 1)$ where $\mathcal{G}_{D,a}$ is the Green function of the Dirichlet Problem and $\mathcal{P}_{D,a}$ the corresponding Poisson type integral. The integral operators are bounded in the Sobolev spaces:

$$\mathcal{P}_{D,a}:H^{s-1/2}(\partial D)\to H^s(D),\quad s\in\mathbb{Z},$$

$$\mathcal{G}_{D,a}:H^{s-2}(D)\to H^s(D)\,(s\geqslant 2),\quad \mathcal{G}_{D,a}:H(D,|\cdot|_{s-2})\to H^s(D)\,(s\leqslant 1)$$

(see, for instance, [16], [17, Theorem 2.26]). In particular, $\mathcal{P}_{D,0}$ is the classical Poisson integral of the Dirichlet Problem for the Laplace operator.

Then for every $v\in C^\infty(\partial D,E)$ the vector-function $g_v=\mathcal{P}_{D,0}\left(\sum\limits_{j=1}^n\frac{a_j}{|\nabla\rho|}\frac{\partial\rho}{\partial x_j}v\right)$ belongs to $C^\infty(\overline{D},E_1)$ and, due to (1), satisfies

$$\tilde\nu_0(g_v)=v \text{ on } \partial D,\quad \|g_v\|_{H^1(D,E_1)}\leqslant C\;\|v\|_{H^{1/2}(\partial D,E)}$$

with a constant $C>0$ being independent on v. Therefore (5) implies, with $\psi=g_v$,

$$\int_{\partial D}v^*(x)v(x)ds(x)=0,$$

i.e. $v=0$ on ∂D.

Further,

$$\|v\|_{H^{-1/2}(\partial D,E)}=\sup_{\phi\neq0}\frac{\left|\int_{\partial D}\phi^*(x)v(x)ds(x)\right|}{\|\phi\|_{H^{1/2}(\partial D,E)}}=$$

$$=\sup_{\phi\neq0}\frac{\left|\int_{\partial D}\tilde\nu_0(g_\phi)^*(x)v(x)ds(x)\right|}{\|g_\phi\|_{A^*,D}}\frac{\|g_\phi\|_{A^*,D}}{\|\phi\|_{H^{1/2}(\partial D,E)}}\leqslant$$

$$\leqslant\|v\|_{A,\partial D}C_{A^*}\sup_{\phi\neq0}\frac{\|g_\phi\|_{H^1(D,E_1)}}{\|\phi\|_{H^{1/2}(\partial D,E)}}\leqslant C_{A^*}C\|v\|_{A,\partial D},$$

i.e. $c_1=(C_{A^*}C)^{-1}$.

On the other hand, by Green formula (4),

$$\|v\|_{A,\partial D}=\sup_{C^\infty(\overline{D},E_1)\ni\psi\neq0}\frac{\left|(A\mathcal{P}_{D,0}(v),\psi)_{L^2(D,E_1)}-(\mathcal{P}_{D,0}(v),A^*\psi)_{L^2(D,E)}\right|}{\|\psi\|_{A^*,D}}\leqslant$$

$$\leqslant\|\mathcal{P}_{D,0}(v)\|_{A,D}\leqslant\|\mathcal{P}_{D,0}(v)\|_{H^1(D,E)}\leqslant c_2\|v\|_{H^{1/2}(\partial D,E)},$$

where positive constant c_2 does not depend on v because the Dirichlet Problem (6) is normally solvable on the Sobolev scale.

Finally, taking $\psi = \nabla \mathcal{P}_{D,1}(v)$ and using Green formula (4) we obtain:

$$\|v\|_{\nabla,\partial D} \geqslant \frac{\left|(\nabla \mathcal{P}_{D,1}(v), \nabla \mathcal{P}_{D,1}(v))_{L^2(D,E_1)} + (\mathcal{P}_{D,1}(v), \Delta \mathcal{P}_{D,1}(v))_{L^2(D)}\right|}{\|\nabla \mathcal{P}_{D,1}(v)\|_{\nabla^*,D}} =$$

$$= \|\mathcal{P}_{D,1}(v)\|_{H^1(D)} \geqslant c_3 \|\tau_0(\mathcal{P}_{D,1}(v))\|_{H^{1/2}(\partial D)} = c_3 \|v\|_{H^{1/2}(\partial D)}$$

with a positive constant c_3 being independent on v because the trace operator $\tau_0 : H^1(D) \to H^{1/2}(\partial D)$ is continuous. □

The completion of $C^\infty(\partial D, E)$ with respect to $\|\cdot\|_{A,\partial D}$ we denote $B_A(\partial D)$.

Theorem 1. *The trace operator τ_0 continuously maps $H_A(D)$ onto $B_A(\partial D)$. In particular, for each $v \in B_A(\partial D)$ the vector $\mathcal{P}_{D,1}(v)$ belongs to $H_A(D)$, satisfies $\tau_0(\mathcal{P}_{D,1}(v)) = v$ on ∂D and*

$$\|\mathcal{P}_{D,1}(v)\|_{A,D} = \|v\|_{A,\partial D} \text{ for all } v \in B_A(\partial D).$$

Proof. Indeed, for every $u \in C^\infty(\overline{D}, E)$ we obtain with the use of Green formula (4)

$$\|\tau_0(u)\|_{A,\partial D} = \sup_{C^\infty(\overline{D},E_1) \ni \psi \neq 0} \frac{\left|(Au, \psi)_{L^2(D,E_1)} - (u, A^*\psi)_{L^2(D,E)}\right|}{\|\psi\|_{A^*,D}} \leqslant \|u\|_{A,D}. \tag{7}$$

Therefore for each $u \in H_A(D)$ we may define the trace as the limit $\tau_0(u) := \lim_{\nu\to\infty} \tau_0(u_\nu)$ in the space $B_A(\partial D)$ where $\{u_\nu\} \subset C^\infty(\overline{D}, E)$ is a sequence approximating u in $H_A(D)$. Estimate (7) implies that the operator $\tau_0 : H_A(D) \to B_A(\partial D)$, defined in this way, is linear and bounded.

Further, for each $v \in C^\infty(\partial D, E)$ we have $\tau_0(\mathcal{P}_{D,1}(v)) = v$ on ∂D and

$$\|v\|_{A,\partial D} \geqslant \frac{\left|(A\mathcal{P}_{D,1}(v), A\mathcal{P}_{D,1}(v))_{L^2(D,E_1)} + (\mathcal{P}_{D,1}(v), \Delta \mathcal{P}_{D,1}(v))_{L^2(D,E)}\right|}{\|A\mathcal{P}_{D,1}(v)\|_{A^*,D}} = \|\mathcal{P}_{D,1}(v)\|_{A,D}.$$

Combining with (7) we easily obtain that $\|\mathcal{P}_{D,1}(v)\|_{A,D} = \|v\|_{A,\partial D}$ for all $v \in C^\infty(\partial D, E)$. Finally, if $v \in B_A(\partial D)$ and $\{v_\nu\} \subset C^\infty(\partial D, E)$ is a sequence approximating v in $B_A(\partial D)$ then the sequence $\{\mathcal{P}_{D,1}(v_\nu)\} \subset C^\infty(\overline{D}, E)$ is fundamental in $H_A(D)$. Hence it converges to a vector function $w \in H_A(D)$ and, by the very definition of the trace, $\tau_0(w) = v$. Clearly, we may interpret w as $\mathcal{P}_{D,1}(v)$ in the sense of the strong extension of the operator $\mathcal{P}_{D,1}$. □

Corollary 1. *The Poisson type integral $\mathcal{P}_{D,1}$ induces an isomorphism $B_A(\partial D) \cong \dfrac{H_A(D)}{H^1_0(D,E)}$.*

Besides, Theorem 1 implies that $H^{1/2}(\partial D, E) \subset B_A(\partial D) \subset H^{-1/2}(\partial D, E)$.

3. The Cauchy Problem

In order to study the Cauchy problem we need one more type of boundary spaces. Let Γ be an open (in the topology of ∂D) connected set of ∂D. Denote by $H_{A,\Gamma}(D)$ the closure of $C^\infty_{\overline{\Gamma}}(\overline{D}, E)$ in $H_A(D)$. Then the differential operator A induces a bounded linear operator $A_\Gamma : H_{A,\Gamma}(D) \to L^2(D, E_1)$ with $\|A_\Gamma\| \leqslant 1$.

According to [7, Lemma 1.2], $H_{A,\Gamma}(D) \subset H^1_{loc}(D \cup \Gamma, E)$. Moreover, [7, Theorem 1.4] states that $H_{A,\Gamma}(D)$ coincides with the set of elements in $H_A(D)$ having zero traces on Γ, i.e. satisfying

$$(Au, g)_{L^2(D,E_1)} - (u, A^*g)_{L^2(D,E)} = 0 \text{ for all } g \in C^\infty(\overline{D}, E_1) \text{ with } \tilde{\nu}_0(g) = 0 \text{ on } \partial D \setminus \Gamma.$$

Following [7], it is natural to define the space of traces on Γ for elements from $H_A(D)$ as the factor space $\dfrac{H_A(D)}{H_{A,\Gamma}(D)}$. However, similar to the situation $\Gamma = \partial D$ above, we want to characterize this space rather in terms of boundary functions.

With this aim we define $B_{A,\Gamma}(\partial D)$ as the closure of $C^\infty_{\overline{\Gamma}}(\partial D, E)$ in $B_A(\partial D)$. Then let $B_A(\overline{\Gamma})$ be the factor space of $B_A(\partial D)$ over $B_{A,\Gamma}(\partial D)$. It is well-known that $B_A(\overline{\Gamma})$ is a normed space. By the very definition every its element extends from Γ up to an element of $B_A(\partial D)$.

Remark 1. *Theorem 1 imply that the norm $\|\cdot\|_{A,\partial D}$ satisfy the parallelogram identity and hence it is a Hilbert space (with the standard scalar product coherent with the norm). This means that $B_A(\overline{\Gamma})$ is actually the orthogonal complement of $B_{A,\Gamma}(\partial D)$ within $B_A(\partial D)$. Then for every $u_0 \in B_A(\overline{\Gamma})$ there is a canonical representative $\tilde{u}_0 \in B_A(\partial D)$ satisfying $\|\tilde{u}_0\|_{A,\partial D} = \|u_0\|_{A,\overline{\Gamma}}$.*

Corollary 2. *The Poisson type integral $\mathcal{P}_{D,1}$ induces an isomorphism $B_A(\overline{\Gamma}) \cong \dfrac{H_A(D)}{H_{A,\Gamma}(D)}$.*

Proof. Indeed, it follows from properties of the Poisson type integral $\mathcal{P}_{D,1}$ that for every $v \in C^\infty_{\overline{\Gamma}}(\partial D, E)$ we have $\mathcal{P}_{D,1}(v) \in C^\infty_{\overline{\Gamma}}(\overline{D}, E)$. Then Theorem 1 implies $B_{A,\Gamma}(\partial D) \overset{\mathcal{P}_{D,1}}{\cong} \dfrac{H_{A,\Gamma}(D)}{H^1_0(D,E)}$. Finally, the desired statement follows from Corollary 1. □

In particular, it follows from Corollary 1 that there is correctly defined continuous linear trace operator $\tau_{0,\Gamma} : H_A(D) \to B_A(\overline{\Gamma})$ and we can easily formulate the Cauchy problem.

Problem 1. *Given $g \in H_{A_1}(D)$ and $u_0 \in B_A(\overline{\Gamma})$ find $w \in H_A(D)$ with*

$$\begin{cases} Aw = g \text{ in } D, \\ \tau_{0,\Gamma}(v) = u_0 \text{ on } \Gamma, \end{cases}$$

i.e.

$$(w, A^*\psi) = (g, \psi) - \langle u_0, \tilde{\nu}_0(\psi)\rangle_{\partial D} \text{ for all } \psi \in C^\infty(\overline{D}, E_1) \text{ with } \tilde{\nu}_0(\psi) = 0 \text{ on } \partial D \setminus \Gamma. \quad (8)$$

Since the time of Hadamard [18] this problem is known to be ill-posed. If $\Gamma \neq \emptyset$ then it has no more than one solution, see, for instance, [5, theorem 10.3.5]. It was actively studied for various types of Dirac operators in various functional spaces [3, 19–24].

We want to reduce Problem 1 to an operator equation of first type in Hilbert spaces. However Theorem 1 implies that Problem 1 can be easily reduced to the following one.

Problem 2. *Given $f \in H_{A_1}(D)$ find $u \in H_{A,\Gamma}(D)$ with $A_\Gamma u = f$.*

Obviously, the reduction of Problem 1 to Problem 2 can be made via

$$f = h - A\mathcal{P}_{D,1}(\tilde{u}_0), \quad u = w - \mathcal{P}_{D,1}(\tilde{u}_0),$$

where $\tilde{u}_0 \in B_A(\partial D)$ is the canonical representative from the class $u_0 \in B_A(\overline{\Gamma})$.

In order to obtain solvabilty conditions for Problem 2 we denote by $A^\star_\Gamma$ the adjoint for A_Γ in the sense of Hilbert spaces. Let also $\mathcal{H}_\Gamma(D)$ be the closed subspace in $H_{A_1}(D)$ consisting of elements h satisfying $A^*h = 0$ in D, $\tilde{\nu}_0(h) = 0$ on $\partial D \setminus \Gamma$, i.e

$$(h, A\phi)_{L^2(D,E_1)} = 0 \text{ for all } \phi \in C^\infty_{\overline{\Gamma}}(\overline{D}, E),$$

and $A_1 h = 0$ in D, $\tau_1(h) = 0$ on Γ, i.e.

$$(h, A_1^*\psi)_{L^2(D,E_1)} = 0 \text{ for all } \psi \in C^\infty_{\partial D\setminus\Gamma}(\overline{D}, E_2).$$

Recall that the adjoint operator for A is a bounded linear map $A^\star_\Gamma : H_{A_1}(D) \to H_{A,\Gamma}(D)$ satisfying

$$(Au, g)_{L^2(D,E_1)} = (Au, AA^\star_\Gamma g)_{L^2(D,E_1)} + (u, A^\star_\Gamma g)_{L^2(D,E)} \text{ for all } u \in H_{A,\Gamma}(D),\ g \in H_{A_1}(D). \quad (9)$$

Theorem 2. *Problem 2 is solvable if and only if 1)* $A_1 f = 0$ *in* D, $\tau_1(f) = 0$ *on* Γ*; 2)* $(f,h)_{L^2(D,E_1)} = 0$ *for all* $h \in \mathcal{H}_\Gamma(D)$*; 3) the series* $u(f) = \sum_{\nu=0}^{\infty} (I - A^\star_\Gamma A_\Gamma)^\nu A^\star_\Gamma f$ *converges in the space* $H_A(D)$*. Moreover, under conditions 1)–3), the series* $u(f)$ *is the unique solution to Problem 2.*

Proof. According to [7, Lemma 2.4] the space $\mathcal{H}_\Gamma(D)$ coincides with kernel of the operator $A^\star_\Gamma$. Then the desired statement follows from [26, Corollary 2.10] because $\|A^\star_\Gamma\| = \|A_\Gamma\| \leqslant 1$. □

Thus, solving Problem 2 we need to identify the operator $A^\star_\Gamma$.

4. Mixed Problems for Helmholtz Equation

In order to identify $A^\star_\Gamma g$ we note that (since $C^\infty_{\overline{\Gamma}}(\overline{D},E) \subset H_A(D)$) equation (9) can be interpreted as a mixed problem for Helmholtz type equation:

$$\begin{cases} A^\star_\Gamma g - \Delta I_k (A^\star_\Gamma g) = A^* g & in \quad D, \\ \tau_0(A^\star_\Gamma g) = 0 & on \quad \Gamma, \\ \tilde{\nu}_0(A(A^\star_\Gamma g) - g) = 0 & on \quad \partial D \setminus \overline{\Gamma}. \end{cases} \tag{10}$$

We consider a little different type of problems with a parameter $a \in \mathbb{R}$.

Problem 3. *Given triple* (h, u_0, u_1) *of vector-distributions find a vector-distribution* w *such that in a proper sense*

$$\begin{cases} a^2 w - \Delta I_k w = h & in \quad D, \\ \tau_0(w) = u_0 & on \quad \Gamma, \\ \tilde{\nu}_0(Aw) = u_1 & on \quad \partial D \setminus \overline{\Gamma}. \end{cases}$$

This type of problems are usually called Zaremba Problems (cf. [27], [28] for $A = \nabla$; in this case $\tilde{\nu}_0(Aw) = \frac{\partial w}{\partial \nu}$ is the normal derivative with respect to ∂D). However, Problem 3 could be ill-posed for $a = 0$ (cf. [7] regarding the case $A = 2\overline{\partial}$). An Existence and Uniqueness Theorem in the space $H_A(D) \cap H^2_{loc}(D \cup \Gamma)$ was obtained for such type of problems in [7, Theorem 3.2] for natural but rather disappointing class of regular data $h \in L^2(D,E)$, $u_0 \in H^{3/2}(\Gamma,E)$, $u_1 \in H^{1/2}(\partial D \setminus \overline{\Gamma}, E)$ if $a \neq 0$.

Working with a Dirac operator $A = 2\overline{\partial}$ one easily can see that in this case $\tilde{\nu}_0 \circ A$ coincides with the complex normal derivative $\overline{\partial}_\nu$ and the trace $\tilde{\nu}_0(Aw)$ exists on ∂D for $w \in H^1(D)$ if and only if $\Delta w \in H^{-1}(D)$, while, in general, $\Delta w \in H(D, |\cdot|_{-1})$ (see, for instance, [24,25]). We seek for a solution to Problem 3 even in a worse than $H^1(D,E)$ class $H_A(D)$. However Theorem 1 allows us to indicate the right classes for the data-triples and for the solution in the formulation of Problem 3. With this aim we will use more negative norms. Namely, set

$$\|u\|_{H^{-1}_{A,\Gamma}(D)} = \sup_{\phi \in C^\infty_{\overline{\Gamma}}(\overline{D},E)} \frac{|(u,\phi)_{L^2(D,E)}|}{\|\phi\|_{A,D}}.$$

The completion of $C^\infty(\overline{D},E)$ with respect to the norm $\|u\|_{H^{-1}_{A,\Gamma}(D)}$ will be denoted by $H^{-1}_{A,\Gamma}(D)$. Again, similar pairing $\langle \cdot, \cdot \rangle_D$ may be defined for pairs $(u,v) \in H^{-1}_{A,\Gamma}(D) \oplus H_{A,\Gamma}(D)$. Then $H^{-1}_{A,\Gamma}(D)$ is naturally embedded to the strong dual $(H_{A,\Gamma}(D))'$ for $H_{A,\Gamma}(D)$ keeping in the mind the corresponding $L^2(D,E)$-pairing (cf. (3)). As the norm $\|\cdot\|_{H^1(D,E)}$ is not weaker than the norm $\|\cdot\|_{A,D}$, there is a positive constant C with $\|u\|_{H(D,|\cdot|_{-1},E)} \leqslant C \|u\|_{H^{-1}_{A,\Gamma}(D)}$ for all $u \in C^\infty(\overline{D},E)$. Then $H^{-1}_{A,\Gamma}(D) \subset H(D, |\cdot|_{-1}, E)$.

Similarly, for $v \in C^\infty(\partial D, E)$, we set

$$\|v\|_{B_A^{-1}(\partial D \setminus \Gamma)} = \sup_{C^\infty_\Gamma(\overline{D},E)\ni\phi\neq 0} \frac{\big|(v,\phi)_{L^2(\partial D,E)}\big|}{\|\phi\|_{A,D}}.$$

It follows from Theorem 1 that in fact $\|v\|_{B_A^{-1}(\partial D \setminus \Gamma)} = \sup\limits_{C^\infty_\Gamma(\partial D,E)\ni\psi\neq 0} \dfrac{\big|(v,\psi)_{L^2(\partial D,E)}\big|}{\|\psi\|_{B_A(\partial D)}}$. For this reason the completion of $C^\infty(\partial D, E)$ with respect to $\|\cdot\|_{B_A^{-1}(\partial D\setminus\Gamma)}$ will be denoted by $B_A^{-1}(\partial D\setminus\Gamma)$. Again, $B_A^{-1}(\partial D \setminus \Gamma)$ is naturally embedded to the strong dual $(H_{A,\Gamma}(D))'$ for $H_{A,\Gamma}(D)$ keeping in the mind the corresponding $L^2(\partial D, E)$-pairing (cf. (3)).

Now we are ready to introduce the right spaces for solving Problem 3. We denote $H_{A,\Gamma,\Delta}(D)$ and $H_{A,\nu\circ A}(D)$ the completions of $C^\infty(\overline{D}, E)$ with respect to the norms

$$\|w\|_{A,\Gamma,\Delta} = \sqrt{\|w\|^2_{A,D} + \|(a^2 - I_k\Delta)w\|^2_{H^{-1}_{A,\Gamma}(D)}}, \quad \|w\|_{A,\Gamma,\nu\circ A} = \sqrt{\|w\|^2_{A,D} + \|\tilde\nu_0(Aw)\|^2_{B_A^{-1}(\partial D\setminus\Gamma)}}$$

correspondingly.

Theorem 3. *The linear spaces $H_{A,\Gamma,\Delta}(D)$ and $H_{A,\Gamma,\nu\circ A}(D)$ coincide and their norms are equivalent. Besides, an element $w \in H_A(D)$ belongs to the space $H_{A,\Gamma,\Delta}(D)$ if and only if $(a^2 - I_k\Delta)w \in H^{-1}_{A,\Gamma}(D)$.*

Proof. The first part of the statement immediately follows from the Green formula

$$(Aw, Au)_{L^2(D,E_1)} + a^2(w,u)_{L^2(D,E)} = ((a^2 - I_k\Delta)w, u)_{L^2(D,E)} - (\tilde\nu_0(Aw), u)_{L^2(\partial D,E)} \quad (11)$$

being true for all $w \in C^\infty(\overline{D}, E)$, $u \in C^\infty(\overline{D}, E)$.

Further, by the definition, $(a^2 - I_k\Delta)w \in H^{-1}_{A,\Gamma}(D)$ if $w \in H_{A,\Gamma,\Delta}(D)$. The proof of the converse statement is similar to that of [24, Corollary 2]. Namely, if $w \in H_A(D)$ and the vector $h = (a^2 - I_k\Delta)w$ belongs to $H^{-1}_{A,\Gamma}(D) \subset H(D, |\cdot|_{-1}, E)$ then according to [16] (cf. also [17, Theorem 2.26] and the proof of Theorem 1) we have $w = \mathcal{G}_{a,D}h + \mathcal{P}_{a,D}w$, where $\mathcal{P}_{a,D}w \in H_{A,\Gamma,\Delta}(D)$ and $\mathcal{G}_{a,D}h \in H^1_0(D, E)$.

Fix now a sequence $\{h_\nu\} \subset C^\infty(\overline{D}, E)$, approximating h in the space $H^{-1}_{A,\Gamma}(D)$. Then, Green formula (11) imply, for $u \in C^\infty(\overline{D}, E)$,

$$(\tilde\nu_0(A\mathcal{G}_{a,D}h_\nu), u)_{L^2(\partial D,E)} = (h_\nu, u)_{L^2(D,E)} - (A\mathcal{G}_{a,D}h_\nu, Au)_{L^2(D,E_1)} + a^2(\mathcal{G}_{a,D}h_\nu, u)_{L^2(D,E)}. \quad (12)$$

As the operator $\mathcal{G}_{a,D} : H(D, |\cdot|_{-1}, E) \to H^1_0(D)$ is bounded we see that

$$\lim_{\nu\to 0} \|\mathcal{G}_{a,D}h_\nu - \mathcal{G}_{a,D}h\|_{A,D} = 0. \quad (13)$$

Therefore (12), (13) yield that the sequence $\{\tilde\nu_0(A\mathcal{G}_{a,D}h_\nu\} \subset C^\infty(\partial D, E)$ is fundamental in the space $B_A^{-1}(\partial D \setminus \Gamma)$. Hence the sequence $\{\mathcal{G}_{a,D}h_\nu\} \subset C^\infty(\overline{D}, E)$ converges in the space $H_{A,\Gamma,\nu\circ A}(D)$ which coincides with the space $H_{A,\Gamma,\Delta}(D)$ according to the already proved part of the theorem. □

Using Green formulae, it is easy to see that, for a triple $(h, u_0, u_1) \in H_A^{-1}(D) \oplus B_A(\overline{\Gamma}) \oplus B_A^{-1}(\partial D \setminus \Gamma)$, Problem 3 in the space $H_{A,\Gamma,\Delta}(D)$ can be interpreted in the following weak sense:

$$(Aw, \psi)_{L^2(D,E_1)} - (w, A^*\psi)_{L^2(D,E)} = \langle u_0, \tilde\nu_0(\psi)\rangle_{\partial D} \text{ for all } \psi \in C^\infty(\overline{D}, E),\ \tilde\nu_0(\psi) = 0 \text{ on } \partial D \setminus \Gamma, \quad (14)$$

$$(Aw, A\phi)_{L^2(D,E_1)} + a^2(w,\phi)_{L^2(D,E)} = \langle h, \phi\rangle_D - \langle u_1, \phi\rangle_{\partial D} \text{ for all } \phi \in C^\infty_\Gamma(\overline{D}, E). \quad (15)$$

Corollary 3. *Let $a \neq 0$. Then for every triple $(h, u_0, u_1) \in H^{-1}_A(D) \oplus B_A(\overline{\Gamma}) \oplus B^{-1}_A(\partial D \setminus \Gamma)$ there is a unique solution $w \in H_{A,\Gamma,\Delta}(D)$ to Problem 3 in the weak sense of (14) and (15). Moreover, there are positive constants c_1, c_2, c_3 such that*

$$\|w\|_{A,\Gamma,\Delta} \leqslant c_1 \|u_0\|_{B_A(\overline{\Gamma})} + c_2 \|u_1\|_{B^{-1}_A(\partial D\setminus\Gamma)} + c_3 \|h\|_{H^{-1}_{A,\Gamma}(D)} \tag{16}$$

Proof. Indeed, fix the canonical representative $\tilde{u}_0 \in B_A(\partial D)$ of the datum $u_0 \in B_A(\overline{\Gamma})$. Then the potential $\mathcal{P}_{D,a}(\tilde{u}_0)$ belongs to $H_{A,\partial D,\Delta}(D) \subset H_{A,\Gamma,\Delta}(D)$ and it coincides with u_0 on ∂D (see Theorem 1). Hence, because of Theorem 3, there is a trace $\tilde{\nu}_0(A\mathcal{P}_{D,a}(\tilde{u}_0)) = v_1$ belonging to $B^{-1}_A(\partial D \setminus \Gamma)$. Therefore mixed problem (14) and (15) is equivalent to finding $u \in H_{A,\Gamma,\Delta}(D)$ satisfying

$$(Au, A\phi)_{L^2(D,E_1)} + a^2(u,\phi)_{L^2(D,E)} = \langle h, \phi\rangle_D - \langle u_1 - v_1, \phi\rangle_{\partial D} \text{ for all } \phi \in C^\infty_{\Gamma}(\overline{D}, E). \tag{17}$$

Of course $w = u + \mathcal{P}_{D,a}(\tilde{u}_0)$.

If $a \neq 0$ then the Hermitian form $(Au, Av)_{L^2(D,E_1)} + a^2(u,v)_{L^2(D,E)}$ induces a scalar product on the space $H_{A,\Gamma}(D)$ and the corresponding norm is equivalent to the original one.

Further, by the very construction, the right hand side of (17) defines a bounded linear functional on $H_{A,\Gamma}(D)$. Now Riesz Theorem on the general form of a continuous linear functional on Hilbert spaces guarantees the existence of a unique element $u \in H_{A,\Gamma}(D)$ satisfying (17). In particular, (17) implies $(a^2 - I_k\Delta)u = h \in H_{A,\Gamma,\Delta}(D)$ and, using Theorem 3, we conclude that u actually belongs to $H_{A,\Gamma,\Delta}(D)$.

Finally, estimate (16) follows from Banach Theorem on the inverse operator. □

Remark 2. *Note that the right hand side in (15) can be replaced by $F(v)$ with an arbitrary continuous linear functional $F \in (H_A(D))'$. Then again Riesz Theorem guarantees a unique solution $w \in H_A(D)$ to Problem 3 in the weak sense of (14) and (15) if $a \neq 0$. However the element w may have no boundary values of $\tilde{\nu}_0(Aw)$ in the sense of distributions over $\partial D \setminus \Gamma$ (cf. (10) where we do not claim that the trace $\tilde{\nu}_0(A(A^\star_\Gamma g))$ on $\partial D \setminus \Gamma$ exists).*

Remark 3. *We also note that, though Problem 3 is Fredholm (according to Corollary 3), it is not an Elliptic Boundary Problem in the sense of Lopatinski because the Dirichlet boundary system $(\tau_0, \tilde{\nu}_0 \circ A)$ is not coercive in general. Of course, it is coercive if $A = \nabla$ but it is not the case for $A = 2\overline{\partial}$.*

The advantage of using Mixed Problem 3 is the following. One may follow the classic approach to Fredholm Boundary Problems and construct the Green function of the problem, say, Φ_Γ. As usual $\Phi_\Gamma(x,y) = \Phi(x,y) - \gamma(x,y)$, where $\Phi(x,y)$ is the bilateral fundamental solution of the Helmholtz operator $a^2 - \Delta$ and the rest $\gamma(x,y)$ is the solution to the mixed problem

$$\begin{cases} a^2\gamma(x,\cdot) - \Delta I_k \gamma(x,\cdot) = 0 & in \quad D, \\ \tau_0(\gamma(x,\cdot)) = \Phi(x,\cdot) & on \quad \Gamma, \\ \tilde{\nu}_0(A\gamma(x,\cdot)) = \tilde{\nu}_0(A\Phi(x,\cdot)) & on \quad \partial D \setminus \overline{\Gamma}. \end{cases}$$

with the smooth data dependent on the parameter $x \in D$ (see [7] for solving). It is only left to say that the fundamental solution $\Phi(x,y)$ may be taken as $b(|x-y|)$ where $b(r)$ is the solution to the famous ordinary differential Bessel type equation:

$$\left(\left(r\frac{\partial}{\partial r}\right)^2 + (n-2)\left(r\frac{\partial}{\partial r}\right) - a^2r^2 \right) b(r) = 0$$

which is unbounded at the origin if $n \geqslant 2$ (see [7] and [29, Ch. 7, §2]); in particular $b(r) = \dfrac{e^{-\sqrt{-1}ar}}{4\pi r}$ for $n = 3$. Obviously, $b(r) = e^{-\sqrt{-1}ar}$ for $n = 1$.

The research was supported by grants Federal Agency of Education "The Development of Science Potential of High School" 2.1.1/4620 and NSH-7347.2010.1.

References

[1] A.N.Tikhonov, V.Ya.Arsenin, Solutions of ill-posed problems, Nauka, Moscow, 1986 (in Russian).

[2] M.M.Lavrent'ev, V.G.Romanov, S.P.Shishatskii, Some ill-posed problems of mathematical physics and analysis, Nauka, Moscow, 1980 (in Russian).

[3] A.A.Shlapunov, N.Tarkhanov, On the Cauchy problem for holomorphic functions of the Lebesgue class L^2 in a domain, Sib. Math. J., **33**(1992), no. 5, 914–922.

[4] A.A.Shlapunov, N.Tarkhanov, Bases with double orthogonality in the Cauchy problem for systems with injective symbols, *Proc. London Math. Soc.*, **71**(1995), no. 1, 1–52.

[5] N.Tarkhanov, The Cauchy problem for solutions of elliptic equations, Berlin, Akademie Verlag, 1995.

[6] B.-W.Schulze, A.A.Shlapunov, N.Tarkhanov, Green integrals on manifolds with cracks, *Annals of Global Analysis and Geometry*, **24**(2003), 131–160.

[7] A.A.Shlapunov, N.Tarkhanov, Mixed problems with a parameter, *Russ. J. Math. Phys.*, **12**(2005), no. 1, 97–124.

[8] N.N. Bogolyubov, D.V. Shirkov, Introduction to the Theory of Quantum fields, Nauka, Moscow, 1984.

[9] B.Booß-Bavnbek, K.P.Wojciechiwski Elliptic Boundary Problems for Dirac operators, Birkhauser, Berlin, 1993.

[10] A.V.Romanov, Convergence of iterations of the Martinlli-Bochner operator and the Cauchy-Riemann equation, *Dokl. AN SSSR*, **242**(1978), no. 4, 780–783.

[11] M.Nacinovich, A.A.Shlapunov, On iterations of the Green integrals and their applications to elliptic differential complexes, *Math. Nachr.*, **180**(1996), 243–286.

[12] V.P.Palamodov, Linear partial differential operators with constant coefficients, Nauka, Moscow, 1967 (in Russian).

[13] K.O.Friedrichs, The identity of weak and strong extensions of differential operators, *Trans. AMS*, **55**(1944), no. 1, 132–151.

[14] Yu.V.Egorov, M.A.Shubin, Linear Partial Differential equations. Foundations of Classical Theory, Results of Science and Technics. Modern problems of Mathematics. Fundamental directions, Moscow, VINITI, **30**(1988).

[15] R.A.Adams, Sobolev Spaces, Academic Press. Inc., Boston et al., 1978.

[16] M. Schechter, Negative norms and boundary problems, Ann. Math., **72** (1960), no. 3, 581–593.

[17] A.A.Shlapunov, N.Tarkhanov, Duality by reproducing kernels, *International Jour. of Math. and Math. Sc.* **2003**(2003), no. 6, 327–395.

[18] J.Hadamard, Le problème de Cauchy et les equations aux derivées partielles linéares hyperboliques, Paris, Gauthier–Villars, 1932.

[19] T.Carleman, Les Fonctions Quasianalytiques, Gauthier–Villars, Paris, 1926.

[20] L.A.Aizenberg, Carleman Formulas in Complex Analysis. First Applications, Nauka, Novosibirsk, 1990 (in Russian).

[21] L.A.Aizenberg, A.M.Kytmanov, On possibility of holomorphic extension to a domain of functions, given on a part of its boundary, *Mat. Sb.*, **182**(1991), no. 5, 490–597.

[22] D.P.Fedchenko, A.A.Shlapunov, On the Cauchy problem for multi-dimensional Cauchy-Riemann operator in Lebesgue L^2 in a domain, *Mat. Sb.*, **199**(2008), no. 11, 141-160.

[23] I.V.Shestakov, On the Cauchy Problem in Sobolev spaces for Dirac operators, *Izv. VUZ. Mathematics*, 2009, no. 7, 51–64.

[24] I.V.Shestakov, A.A.Shlapunov, Negative Sobolev Spaces in the Cauchy Problem for the Cauchy-Riemann Operator, *J. SFU. Mathematics and Physics*, 2009, no. 1, 17–30.

[25] I.V.Shestakov, A.A.Shlapunov, On the Cauchy problem for operators with injective symbols in spaces of distributions, *J. Inverse and Ill-posed Problems*, **19**(2011), no. 1, 43-54.

[26] A.A.Shlapunov, On iterations of non-negative operators and their applications to elliptic systems, Math. Nachr., 2000, no. 218, 165–174.

[27] S.Zaremba, Sur un problème mixte à l'équation de Laplace, Bull. Intern. Acad. Sci. Cracovie (1910), 314–344.

[28] S.Simanca, Mixed elliptic boundary value problems, *Comm. in PDE*, **12**(1987), 123–200.

[29] A.N.Tikhonov, A.A.Samarskii, Equations of Mathematical Physics, Nauka, Moscow, 1972 (in Russian).

Граничные задачи для уравнения Гельмгольца и задача Коши для операторов Дирака

А.А. Шлапунов

При изучении операторного уравнения $Au = f$ в пространствах Гильберта обычно требуется знать сопряженный $A^\star$ оператор для A. Решая некорректную задачу Коши для операторов типа Дирака в пространствах Лебега одним итерационным методом, мы предлагаем построить соответствующий сопряженный оператор при помощи нормально разрешимой смешанной задачи для уравнения Гельмгольца. Это ведет к описанию условий разрешимости задачи Коши и к построению ее точного и приближенных решений.

Ключевые слова: смешанная задача, уравнение Гельмгольца, операторы Дирака, некорректная задача Коши.