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On Strongly Algebraically Closed Lattices

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In this article some fundamental properties of existentially and algebraically closed lattices are investigated. We also define the notion of strongly algebraically closed lattices and we show that a q'-compact complete boolean lattice is strongly algebraically closed.

Keywords: existentially and algebraically closed lattices, strongly algebraically closed lattices, equationally noetherian lattice, complete Boolean algebras.
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Introduction

Suppose $\mathcal{L}$ is an algebraic language and A is an algebra of type $\mathcal{L}$. If we attach the elements of A as constants to $\mathcal{L}$, then the new language will be denoted by $\mathcal{L}(A)$. Let $\mathfrak{X}$ be a class of algebras of type $\mathcal{L}$ and $A \in \mathfrak{X}$. We say that A is existentially closed in the class $\mathfrak{X}$, if every existential sentence φ in the language $\mathcal{L}(A)$ which is true in some extension $A \subseteq B \in \mathfrak{X}$, is also true in A. If we restrict ourself to the positive existential sentences, then we obtain the definition of an algebraically closed algebra in the class $\mathfrak{X}$. Equivalently, A is existentially closed in $\mathfrak{X}$, if and only if every finite set of equations and inequations with coefficients from A, which is solvable in some $B \in \mathfrak{X}$ containing A, already has a solution in A itself. Similarly A is algebraically closed in $\mathfrak{X}$, if and only if every finite set of equations with coefficients from A, which is solvable in some $B \in \mathfrak{X}$ containing A, already has a solution in A.

Many articles have already been published concerning the existentially and algebraically closed algebras in several algebraic structures. In a preprint, M. Shahryari investigated some applications of these concepts to special classes of groups (see [5]). In this paper, we focus on distributive lattices. Already, in [4] J. Schmid proved that in the class of distributive lattices, a lattice A is algebraically closed if and only if, it is boolean. Also, he showed that A is existentially closed, if and only if, it is an atomless boolean lattice (see [4]). Schmid asked about the situation in which a distributive lattice is strongly algebraically closed. Note that we say that an algebra A is strongly algebraically closed in a class $\mathfrak{X}$, if every set of equations (finite or infinite) with coefficients from A, which is solvable in some $B \in \mathfrak{X}$ containing A, already has a solution in A. In the case of lattices, it is easy to see that such a distributive lattice must be a complete boolean lattice. In this paper, we define the notion of a q'-compact lattice and we show that if such a lattice is also complete boolean, then it is strongly algebraically closed. The organization of the paper is as follows: in Section 1, we show that if a class $\mathfrak{X}$ of lattices is inductive and closed under elementary sublattices, then every element of $\mathfrak{X}$ has an extension which is existentially closed in $\mathfrak{X}$. In fact, this result is not new and at least a version of it for classes of groups is presented in [5]. However, in our version, the assumption of being closed under elementary substructures is applied instead of the stronger hypothesis of being closed under substructures.

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In Section 2, we collect some elementary properties of algebraic and existentially closed lattices in some small classes. We also obtain a connection between variants of the property of being equationally noetherian and completeness of distributive lattices. Section 3 consists of a review of some notions of unltraproduct construction and universal algebraic geometry. The last section contains a proof of the main theorem. The reader can consult any standard text of universal algebra and model theory for basic notions of logic, and also any book of lattice theory can be used for elementary definitions of lattices. We will assume that the reader is familiar with free lattices, boolean lattices and algebras and the possibility of extending any distributive lattice to a boolean one.

1. Existentially closed lattices

For lattices A and B we write $A \leqslant B$ if A is a sublattice of B and refer to this as an extension of H. An equation with parameters (or coefficients) in lattice A is an expression of the form

$$f(a_1, \ldots, a_m, x_1, \ldots, x_n) \approx g(a_1, \ldots, a_m, x_1, \ldots, x_n)$$

where f, g are lattice polynomials and $a_1, \ldots, a_m \in A$ are the parameters. Replacing $\approx$ by $\not\approx$, we obtain an inequality (with parameters in A). A finite system of equations and inequations with parameters in A has the form:

$$S = \{f_1 \approx g_1, \ldots, f_m \approx g_m, f_{m+1} \not\approx g_{m+1}, \ldots, f_{m+n} \not\approx g_{m+n}\}.$$

Note that expressions of the form $f \leqslant g$ or $f < g$ may be replaced by $f \wedge g = f$ or $f \wedge g = f, f \not\approx g$, respectively, so there is no need to include such expressions in our considerations. A lattice A is called existentially closed, if any finite consistent system of equations and inequations with coefficients from A, has a solution in A. A system S with coefficients in A is called consistent, if there is an extension B, such that S has a solution in B. One can generalize this definition to an arbitrary class of lattices. Let $\mathfrak{X}$ be a class of lattices. A lattice $A \in \mathfrak{X}$ is called existentially closed in the class $\mathfrak{X}$, if every $\mathfrak{X}$-consistent system S has a solution in A. Here, $\mathfrak{X}$-consistency means that there exists a lattice $B \in \mathfrak{X}$ which contains A and S has a solution in B. Recall that a class of lattices is called inductive, if it contains the union of any chain of its elements. A version of the following lemma which uses of the assumption of being closed under substructure is known at least for classes of groups. But we give a proof for the lattice case with a weaker assumption of being closed under elementary substructure. Also note that, sorts of the lemma and the forthcoming theorem are standard in model theory for elementary classes of first order structures, but the classes used in our versions are not necessarily axiomatizable.

Lemma 1.1. *Suppose that $\mathfrak{X}$ is an inductive class of lattices which is closed under the operation of taking elementary sublattices. Let $A \in \mathfrak{X}$. Then there is $B \in \mathfrak{X}$ which $A \subseteq B$ and $|B| \leqslant \max\{\aleph_0, |A|\}$. Also, for any finite system of equations and inequations S over A, we have either of the following assertions:*
(1) S has a solution in B.
(2) For any extension $B \subseteq C \in \mathfrak{X}$, the system S has no solution in C.

Proof. Let $\kappa = \max\{\aleph_0, |A|\}$. It is clear that the cardinal of the set of all systems of equations and inequations with coefficient A is at most κ. So, we can suppose $\{S_\alpha\}$ is the set of all such systems indexed by ordinals $0 \leqslant \alpha \leqslant \kappa$. Let $A_0 = A$ and suppose that $A_\gamma \in \mathfrak{X}$ is previously defined such that $|A_\gamma| \leqslant \kappa$ and

$$\beta < \gamma \Longrightarrow A_\beta \subseteq A_\gamma.$$

Suppose that $K_\alpha = \cup_{\gamma<\alpha} A_\gamma$. Since $\mathfrak{X}$ is inductive, so $K_\alpha \in \mathfrak{X}$, also

$$|K_\alpha| \leqslant \sum_{\gamma<\alpha} |A_\gamma| \leqslant \kappa^2 = \kappa.$$

If S_α has no solution in any extension of K_α, then we let $A_\alpha = K_\alpha$. If there is $C \in \mathfrak{X}$ such that $K_\alpha \subseteq C$ and S_α has a solution $\overline{u} = (u_1, \ldots, u_n)$ in C we let A_α to be the least elementary sublattice of C containing K_α and the solution $\overline{u} = (u_1, \ldots, u_n)$. Note that by Lowenheim-Skolem theorem, we may assume that A_α has the cardinality at most κ. Recall that $\mathfrak{X}$ is closed under elementary sublattice, consequently $A_\alpha \in \mathfrak{X}$. In this way, for all $0 \leqslant \alpha \leqslant k$ the lattice A_α is defined. Also,

$$\beta < \alpha \Longrightarrow A_\beta \subseteq A_\alpha.$$

Now, we put $B = \bigcup A_\alpha \in \mathfrak{X}$. This lattice has the desired properties. Because, if a system $S = S_\alpha$ has no solution in B, then it has no solution in any extension of K_α. Therefore, it has no solution in any extension B. □

Theorem 1.2. *Suppose that $\mathfrak{X}$ is an inductive class of lattices closed under elementary sublattice and $A \in \mathfrak{X}$. Then there is $A^* \in \mathfrak{X}$ such that:*
(1) $A \leqslant A^*$.
(2) A^* is existentially closed in class $\mathfrak{X}$.
(3) $|A^*| \leqslant \max\{\aleph_0, |A|\}$.

Proof. Assume $A^0 = A$ and $A^1 = B$, where B is the lattice already obtained in Lemma 1.1. Now, suppose that A^m is already defined and A^{m+1} is the lattice which its existence is proved in Lemma 1.1 for $A = A^m$. Let

$$A^* = \cup_m A^m.$$

Since $\mathfrak{X}$ is inductive, so $A^* \in \mathfrak{X}$ and as well A^* satisfies condition (1) and (2). Suppose that S is a consistent system with coefficients from A^*. Since S is finite, so there exists m such that S is system with coefficient in A^m. Therefore, S has a solution in A^{m+1}. □

As an application, suppose that $\mathfrak{X}$ is the class of distributive lattices. Then by the above theorem, every distributive lattice A has an extension B with the cardinality at most $\max\{\aleph_0, |A|\}$, such that B is existentially closed in $\mathfrak{X}$. Since by [4] such a lattice is an atomless boolean lattice, so we conclude that every distributive lattice A can be extended to an atomless boolean lattice of the cardinality at most $\max\{\aleph_0, |A|\}$.

2. Existentially closed lattices in small classes

Suppose A is a sublattice of B. Then the class $\mathfrak{X} = \{A, B\}$ is inductive. In this section we discuss about the situation in which A is algebraically or existentially closed in $\mathfrak{X}$. If this happens, then we simply say that A is algebraically (existentially) closed in B. Recall that a sublattice A of a lattice B is called a retract of B, if there is a homomorphism (termed retraction) $\varphi : B \to A$ which is identity on A. We will show that every retract of B is algebraically closed in B. A similar investigation for the case of groups is done By Miasnikov and Romankov in [3]. Here, we assume that $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set of variables, and $FL[X]$ is the free lattice with basis X.

Definition 2.1. *A lattice A is called equationally noetherian, if any system of equations with coefficient in A is equivalent with a finite subsystem. If any system of equations over A is equivalent with a finite system then it is said weakly equationally noetherian.*

In the following theorem we prove some properties of retracts. This theorem as well as its proof is the lattice version of the similar theorem for groups in [3].

Theorem 2.2. *Let $A \leqslant B$ be a lattice extension. Then the following hold:*
(1) If A is a retract of B, then A is algebraically closed in B.

(2) Suppose that B is finitely presented and A is finitely generated. Then A is algebraically closed in B if and only if A is a retract.

(3) Suppose that B is finitely generated relative to A and A is equationally noetherian. Then A is algebraically closed in B if and only if A is a retract.

Proof. Suppose that A is a retract of B and $\varphi : B \to A$ is a retraction. Let

$$S = \{f_1 \approx g_1, \dots, f_m \approx g_m\}$$

be a finite system of equations that has a solution $b_1, \dots, b_n$ in B. Then one can check that S has a solution $\varphi(b_1), \dots, \varphi(b_n)$ in A. This shows that A is algebraically closed in B.

To prove (2), assume that A is generated by a finite set $a_1, \dots, a_m$ and B is a finitely presented as $B = \langle b_1, \dots, b_n \,| r_1 \approx s_1, \dots, r_k \approx s_k\rangle$. Every a_i can be expressed as a lattice word in letters $b_1, \dots, b_n$, so assume that $a_i = \varphi_i(b_1, \dots, b_n)$, for $i = 1, \dots, m$. Consider the system of equations

$$\varphi_1(x_1, \dots, x_n) \approx a_1, \dots, \varphi_m(x_1, \dots, x_n) \approx a_m,$$

$$r_1(x_1, \dots, x_n) \approx s_1(x_1, \dots, x_n), \dots, r_k(x_1, \dots, x_n) \approx s_k(x_1, \dots, x_n)$$

with the constants $a_i \in A$. This system has a solution

$$x_1 = b_1, \dots, x_n = b_n$$

in B. Hence the system has a solution

$$x_1 = u_1, \dots, x_n = u_n$$

in A. Now, we can define the map $q : b_1 \to u_1, \dots, b_n \to u_n$. Since $(u_1, \dots, u_n)$ satisfies the defining relations of B, so this map is a lattice homomorphism. Furthermore, we have $q(a_i) = q(\varphi_i(b_1, \dots, b_n)) = \varphi_i(q(b_1), \dots, q(b_n)) = \varphi_i(u_1, \dots, u_n) = a_i$. This shows that q is a retraction. Statement (3) is similar to (2). □

We say that a lattice extension $A \leqslant B$ is discriminating, if for any finite set W of elements from B, there exists a retraction $h : B \to A$ whose restriction into W is injective. The following theorem gives some equivalences of being existentially closed in a lattice B. It is a lattice form of a similar theorem from [3] and so we don't give any proof.

Theorem 2.3. *Let $A \leqslant B$ be a lattice extension. Then the following hold:*

(1) If $A \leqslant B$ is discriminating, then the sublattice A is existentially closed in B.

(2) Suppose that B is finitely generated relative to A and A is equationally noetherian. Then A is existentially closed in B if and only if the extension $A \leqslant B$ is discriminating.

3. Existentially closed lattices in class of distributive lattices

In [4] J. Schmid studied algebraically closed and existentially closed distributive lattices. He proved that a distributive lattice is algebraically closed if and only if it is a boolean lattice. He showed also that a distributive lattice is existentially closed if and only if it is free of atoms. He worked within the class $\mathfrak{D}$ whose elements are distributive lattices. In [4] J. Schmid shows that any strongly algebraically closed distributive lattice is a complete boolean lattice. The main aim of this article is to prove that a complete boolean lattice having the property of being q'-compactness is strongly algebraically closed.

Definition 3.1. *A lattice A in a class $\mathfrak{X}$ is said strongly algebraically closed if every system (not necessarily finite) of equations with parameters in A which has a solution in some extension $B \in \mathfrak{X}$, has already a solution in A.*

We will need the compactness theorem of Godel-Malcev and so we give a short review of this theorem. Recall that a set of formulas T in a language $\mathcal{L}$ is called satisfiable in a class $\mathfrak{X}$ (or T is realized in $\mathfrak{X}$), if one can assign some elements from a particular algebraic structure from $\mathfrak{X}$ as values to the variables which occur in T in such a way that all formulas from T become true. The set T is called finitely satisfiable in $\mathfrak{X}$, if every finite subset of T is realized in $\mathfrak{X}$ (see [1] and [2]).

Theorem 3.2 (Compactness Theorem)**.** *If a set of first-order formulas T is finitely satisfiable in a class $\mathfrak{X}$, then T has a solution in an ultraproduct of structures from $\mathfrak{X}$.*

As an special case, we focus on the case of lattices and systems of equations. Let $\mathfrak{X}$ be a class of lattices, and S be system of equations of lattices. If every finite subset $S_0 \subseteq S$ has a solution in element of $\mathfrak{X}$, then S has a solution in an ultraproduct of elements of $\mathfrak{X}$.

Corollary 3.3. *Let A be a lattice and S be an arbitrary system of equations over A. If any finite subsystem S has a solution in A, then S has a solution in an ultrapower of A.*

Equationally and weak equationally noetherian boolean algebras (with coefficients) are characterized by Shevlyakov in [6]. In the proof of our main theorem, we will need his results. As is shown in [6], a boolean algebra A is equationally noetherian if and only if it is finite; and it is weak equationally noetherian if and only if it is complete. In what follows, we need a new generalization of the concept of equational noetherian algebra. I am grateful to my supervisor M. Shahryari who introduced this idea to me. Let X be a possibly infinite set of variables. Let A be an algebra in a language $\mathcal{L}$. A system S of equations in language $\mathcal{L}(A)$ and variables in X is said to be finitary, if it contains finite number of variables. Let S and S' be two finitary systems and suppose that the sets of variables occurring in S and S' are respevtively $\{x_1, \ldots, x_n\}$ and $\{x_1, \ldots, x_m\}$. Let also $n \leqslant m$. We say that S is reducible to S' over A, if and only if, for some extension $B \supseteq A$ we have $V_A(S) = A^n \cap \pi(V_B(S'))$. Here π is the projection $B^m \to B^n$ and $V_A(S)$ and $V_B(S')$ are the sets of solutions of S and S' in A and B, respectively.

We say that the algebra A is finitary equational noetherian, if every finitary system of equations in the language $\mathcal{L}(A)$ is reducible over A to a finite system. The following proposition gives a close connection between completeness of lattices and finiteness conditions for systems of equations.

Proposition 3.4. *Let A be an equationally noetherian lattice. Then A is complete. Conversely, if a distributive lattice A is complete, then it is finitary equational noetherian.*

Proof. Let A be equational noetherian and $K \subseteq A$ be a non-empty set. We show that K has the largest lower bound. Let $L = \{b \in A : b \leqslant K\}$ and consider the following system of equations

$$\begin{aligned} x \wedge a &\approx x \ (a \in K), \\ x \wedge b &\approx b \ (b \in L). \end{aligned}$$

This system is equivalent over A to some finite subsystem

$$\begin{aligned} x \wedge a_1 &\approx x, \\ &\vdots \\ x \wedge a_n &\approx x, \\ x \wedge b_1 &\approx b_1, \\ &\vdots \\ x \wedge b_m &\approx b_m, \end{aligned}$$

where $a_i \in K$ and $b_j \in L$, for all i and j. Suppose $x = a_1 \wedge \cdots \wedge a_n$. Clearly, this x is a solution of the second system in A and hence it is a solution of the first system. This shows that $\bigwedge K = a_1 \wedge \cdots \wedge a_n$, so A is complete.

Now, suppose that A is a complete distributive lattice. As we saw at the end of the Section 2, A can be embedded in a boolean lattice, so suppose that B is the smallest boolean lattice containing A. Clearly B is complete and so, by [6], B is weak equational noetherian in the language $\mathcal{L}'(B)$, where $\mathcal{L}'$ is the language of boolean algebras. Note that every system of equations in the larger language $\mathcal{L}'(B)$ can be transformed to a system of equations in $\mathcal{L}(A)$, by introducing finitely many new variables and finitely many new equations. Let X be a possibly infinite set of variables and S be a finitary system of equations in the language $\mathcal{L}(A)$ and variables $x_1, \ldots, x_n$. We know that there exists a finite system S_0 in the language $\mathcal{L}'(B)$ (and with the same variables) such that $V_B(S) = V_B(S_0)$. Introducing some new variables and new equations in the language $\mathcal{L}(A)$, we obtain a finite system T such that $V_B(S_0) = \pi(V_B(T))$, where π is a projection. Now, we have

$$V_A(S) = A^n \cap V_B(S) = A^n \cap V_B(S_0) = A^n \cap \pi(V_B(T)).$$

This shows that S is reducible to a finite system and so A is finitary equational noetherian. □

Let S be a system of equations in A. The set of all logical consequences of S over A is the radical $Rad_A(S)$. In other words, $Rad_A(S)$ is the set of all lattice equations $f \approx g$ such that $V_A(S) \subseteq V_A(f \approx g)$. We say that two lattices A and B are geometrically equivalent, if for any system S, we have $Rad_A(S) = Rad_B(S)$. We say that a lattice A is q'-compact, if it is geometrically equivalent to any of its elementary extensions. We are now ready to prove our main theorem.

Theorem 3.5. *Let A be complete boolean lattice which is q'-compact. Then A is strongly algebraically closed in the class of distributive lattices.*

Proof. Let $\mathcal{L}'$ be the language of boolean algebras. Note that, in the same time we can consider A as a boolean algebra. Let S be a consistent system in the language $\mathcal{L}(A)$. Clearly, S is also a system in $\mathcal{L}'(A)$. Since A is complete so by [6], it is a weak equational noetherian boolean algebra. So, there is a finite system T in the language $\mathcal{L}'(A)$ equivalent to S over A. We know that every finite $S_0 \subseteq S$ is consistent, and by the result [4] of Schmid, A is algebraically closed. Hence, every such S_0 has a solution in A. By Corollary 3.3, S has a solution in some ultra-power $B = A^I/\mathcal{U}$. Note that B is also a distributive lattice, and since it is an elementary extension of A, and A is q'-compact then

$$Rad_A(S) = Rad_B(S).$$

On the other hand, we have

$$Rad_A(S) = Rad_A(T).$$

Since T is finite, we also have

$$Rad_A(T) = Rad_B(T).$$

This shows that S and T are equivalent over B. Therefore, T has a solution in B and consequently in A. Thus S has a solution in A. Note that according to Proposition 3.4, T is a finite system in the language $\mathcal{L}'(A)$. But by introducing a finite number of new variables and a finite number of new equations we can transform it to a finite system in $\mathcal{L}(A)$. To do this, we perform the following actions:

1. If T contains the boolean constants 0 and 1, then there will be no change, since $0, 1 \in A$.

2. If T contains therm x', then we introduce a new variable y and insert new equations $x \wedge y \approx 0$ and $x \vee y \approx 1$, instead.

3. If there appears a term of the form a', then again there will not be any change. □

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О сильно алгебраически замкнутых решетках

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В этой статье исследованы некоторые фундаментальные свойства экзистенциально и алгебраически замкнутых решеток. Определено также понятие сильно алгебраически замкнутой решетки. Показано, что q'-компактное полная булева решетка сильно алгебраически замкнута.

Ключевые слова: экзистенциально и алгебраически замкнутые решетки, сильно алгебраически замкнутые решетки, эквационально нетерова решетка, полные булевы алгебры.