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On Solvability of Systems of Symbolic Polynomial Equations

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Approaches to solving the systems of non-commutative polynomial equations in the form of formal power series (FPS) based on the relation with the corresponding commutative equations are developed. Every FPS is mapped to its commutative image – power series, which is obtained under the assumption that all symbols of the alphabet denote commutative variables assigned as values in the field of complex numbers. It is proved that if the initial non-commutative system of polynomial equations is consistent, then the system of equations being its commutative image is consistent. The converse is not true in general.

It is shown that in the case of a non-commutative ring the system of equations can have no solution, have a finite number of solutions, as well as having an infinite number of solutions, which is fundamentally different from the case of complex variables.

Keywords: non-commutative ring, polynomial equations, formal power series, commutative image.
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Introduction

Let us consider a system of polynomial equations

$$P_j(z,x)=0,\ j=1,2,\dots,n, \tag{1}$$

which is solved relative to symbols $z=(z_1,\dots,z_n)$ in the form of formal power series (FPS), depending on the symbols $x=(x_1,\dots,x_m)$. One can interpret these symbols as follows.

Firstly, we can consider the symbols $z_1,\dots,z_n,x_1,\dots,x_m$ as an alphabet, over which non-commutative multiplication (concatenation) and commutative operation of the formal sum are determined; besides, the commutative multiplication by complex numbers is defined. The multiplication of any element by (-1) gives the inverse element with respect to addition, so the alphabet generates the ring of symbolic polynomials and FPS with numeric (complex) coefficients.

According to the second interpretation, the symbols $z_1,\dots,z_n,x_1,\dots,x_m$ are treated as variables with values from a ring, in which as usual the operation of addition is commutative, while for the operation of multiplication commutative is not required; it is also assumed that elements

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of the ring may be multiplied by complex numbers. Such a ring is formed, for example, by square matrices of given order over the field of complex numbers. In this case, the problem of solving the system (1) is to express the matrix variables $z_1, \dots, z_n$ in the form of FPS depending on the non-commutative matrix variables $x_1, \dots, x_m$.

The systems of non-commutative polynomial equations have applications. So, in one interpretation, they are grammars that generate certain classes of formal languages. These include context-free languages (cf-languages), languages of immediate constituents (ic-languages), languages in the Greibach normal form and others [1,2]. In formal languages theory the system (1) is considered as a grammar over the terminal symbols $x_1, \dots, x_m$ which form the vocabulary (alphabet) of the language and non-terminal symbols $z_1, \dots, z_n$ needed to specify the grammar rules. In this interpretation the monomials are considered as the sentences (words) of the language, and the FPS, which is the solution of system (1), considered as the formal language generated by the grammar and representing the formal sum of all «correct» sentences [1, 2]. In other interpretation the application of systems of equations over non-commutative ring of matrices associated with the conversion of power series of several matrices [3].

Issues related to solution of the non-commutative system (1) has been little investigated. The main difficulties are connected with the non-commutativeness of multiplication and the absence of division, preventing the elimination of unknowns. In particular, in a non-commutative ring there is no concept of determinant.

Among the few known results are properties of the Chomsky-Schutzenberger system of equations

$$z_j = Q_j(z, x),\ j = 1, 2, \dots, n,$$

on the right-hand sides of which the natural conditions imposed: $Q(0, 0) \neq 0$ and the polynomial $Q_j(z, 0)$ does not contain linear monomials, $j = 1, 2, \dots, n$ [1,2]. In particular, it is known that the Chomsky-Schutzenberger system of equations has a unique solution in the form of FPS, which can be obtained by the method of successive approximations:

$$z^{(k+1)}(x) = Q(z^{(k)}(x), x),\ k = 0, 1, \dots\ ,$$

$$z^{(0)} = 0,$$

$$z(x) = \lim_{k\to\infty} z^{(k)}(x);$$

here $Q(z, x) = (Q_1(z, x), \dots, Q_n(z, x))$.

In formal languages theory Chomsky-Schutzenberger systems of equations are grammars that generate important class of cf-languages and that is of interest to their solution. More precisely, a cf-language is the FPS, representing the first component $z_1(x)$ of the solution of such a system [1,2].

We note that for commutative image of the Chomsky-Schutzenberger system of equations the condition of the implicit map theorem is fulfilled and provides the existence and the uniqueness of a holomorphic at zero commutative solution of this system. However, the solvability of the initial system of equations of Chomsky-Schutzenberger does not follow from here, because it turns out that the commutative image of a non-commutative inconsistent system can be a consistent system (the solvability of the Chomsky-Schutzenberger system of equations is justified by the method of successive approximations).

The purpose of this paper is to obtain solvability conditions of system (1) in terms of the commutative image of this system, which is obtained under the assumption that all variables included in the system take the values from the field of complex numbers.

The commutative image of the system (1) is a system of nonlinear algebraic equations for the study of which it is possible to use methods of complex analysis and algebraic geometry, including those developed for solving of systems which have special form [4]. This article examines the relationship between the solutions of the system (1) and its commutative image: the first step in this direction can be to obtain conditions of consistence and inconsistence for these systems, and also availability of infinite number of solutions.

1. FPS as a solution of the system of symbolic equations

Despite the extensive literature on formal languages theory, it is advisable to preliminarily clarify some definitions related to solving of the non-commutative systems in the form of the FPS. First of all check, what does the equality of two FPS mean, if the variables $x_1, \dots, x_m$ take the values from a ring in which the multiplication is non-commutative.

We assume that the terms of a FPS are ordered as follows: let all monomials from $x_1, \dots, x_m$ be grouped in homogeneous polynomials arranged in ascending order of the degrees, then the monomials each of them are numbered in the lexicographic order, moving from less to greater degree. Thus, the number zero is assigned to the zero degree monomial (unit of the ring), the numbers $1, \dots, m$ are assigned to the monomials $x_1, \dots, x_m$, linear combination of which forms homogeneous polynomial of the first degree, then the monomials of homogeneous polynomial of the second degree are also numbered in the lexicographic order, etc. With this ordering all possible monomials over symbols $x_1, \dots, x_m$ in the unique way be written as the sequence $\{u_i\}_{i=0}^{\infty}$, which plays the role of the universal basis of FPS from $x_1, \dots, x_m$. Now every series s can be uniquely written in the form of expansion in the universal basis:

$$\sum_{i=0}^{\infty} \langle s, u_i \rangle u_i, \tag{2}$$

here $\langle s, u_i \rangle$ is the numeric coefficient of a monomial u_i. Finally, two FPS are said to be equals if and only if its corresponding numerical coefficients of the universal basis monomials are equal.

It is useful to note that, substituting the solution

$$z_1 = \sum_i \langle z_1, u_i \rangle u_i, \ \dots \ , z_n = \sum_i \langle z_n, u_i \rangle u_i \tag{3}$$

of system (1) in the polynomial $P_j(z, x)$ standing on the left side of the equation, we obtain the FPS in the form of an expansion in the universal basis with zero coefficients:

$$l_j = \sum_i \langle l_j, u_i \rangle u_i = \sum_i \ 0 \cdot u_i, \quad j = 1, \dots, n.$$

As a tool to study non-commutative systems, we need the commutative image of a multiple FPS.

2. The commutative image of a FPS and a system of symbolic equations

We will give a short FPS (2) of its commutative image $ci(s)$ that is the power series, which is obtained from s under the assumption that symbols $x_1, \dots, x_m$ denote commutative variables taking values from the field of complex numbers.

Under this assumption any monomial u_i with symbols $x_1, \ldots, x_m$ can be written in the form $x_1^{\alpha_1} \cdot \ldots \cdot x_m^{\alpha_m}$, where α_j is the number of occurrences (degree) $\deg_{x_j}(u_i)$ of symbol x_j in this monomial, $j = 1, \ldots, n$. If we denote multi-index $\alpha = (\alpha_1, \ldots \alpha_m)$, one can write the equality $\alpha = \deg_x(u_i)$, taking into account we get the following equalities:

$$ci(s) = ci\left(\sum_{i=0}^{\infty} \langle s, u_i\rangle u_i\right) = \sum_{i=0}^{\infty} \langle s, u_i\rangle ci(u_i) = \sum_{\alpha}\left(\sum_{\alpha=\deg_x(u_i)} \langle s, u_i\rangle\right) x^\alpha \stackrel{\text{def}}{=} \sum_{\alpha} c_\alpha x^\alpha.$$

For the first time the commutative image of a FPS was considered by A.L. Semenov, who used it as a tool for solving for algorithmic problems associated with the cf-languages [5]. Further the commutative image has found numerous applications, in particular related to the fact that the commutative image of a cf-language is an algebraic function of several complex variables [6–8].

Let us denote $z = z(x) = (z_1(x) \ldots, z_n(x))$ the solution of system (1) presented by series (3). We have the following theorem.

Theorem 2.1. *If $z = z(x)$ is a solution of the non-commutative system of equations (1) in the form of symbolic FPS, then commutative FPS $z = ci(z(x))$ over the field of complex numbers converge in some neighborhood of zero, defining germs of holomorphic algebraic functions, and are solutions of the commutative system of equations*

$$ci(P_1(z, x)) = \ldots = ci(P_n(z, x)) = 0. \tag{4}$$

Proof. We note that with regard to the designations made above, we have the equality:

$$ci(P_j(z, x))|_{z=ci(z(x))} = ci(P_j(z(x), x)) = ci(l_j) = \sum_i 0 \cdot ci(u_i) = 0, \quad j = 1, 2, \ldots, n;$$

i.e. the commutative FPS $z = ci(z(x))$ satisfy the system of equations (4).

Further, we show that all FPS $ci(z(x))$ converge in some neighborhood of zero. Indeed, if the FPS over the field of complex numbers satisfy an equation with holomorphic coefficients (functions represented by the power series which absolutely converge in a neighborhood of zero), then it also converges absolutely in some neighborhood of zero, representing a holomorphic function [9]. It means, the commutative FPS $ci(z(x))$ converge in a neighborhood of zero, giving rise to germs of an algebraic vector function, and the solutions (z, x) of the system (4), considering as points in the complex space $\mathbb{C}^{n+m}_{z,x}$, can be written by using this vector function in the form of $(z(x).x)$. Theorem 1 is proved. □

We emphasize that the consistency, i.e. the existence of a solution, is understood for non-commutative system (1) and commutative one (2) differently: in the first case the solutions are symbolic FPS, and in the second case there are points in the complex space, parameterized by the algebraic holomorphic mapping $z = z(x)$ the holomorphic branches of which are commutative images of the symbolic FPS are components of the solution of the system (1).

3. Consistent and inconsistent systems of symbolic equations

As shown above, if the non-commutative system (1) is consistent, then the commutative system (4) is also consistent. The reverse is not true in general. Indeed, the system of equations

$$\begin{aligned} z_1 - z_2 &= 0, \\ z_1 - z_2 - x_1x_2 + x_2x_1 &= 0 \end{aligned}$$

is inconsistent, whereas its commutative image has infinitely many solutions: $z_1 = z_2 = t$, where t is an arbitrary complex number or a function.

By the way, this example refutes the hypothesis that the system (1) has a solution in the form of a FPS if and only if the system (4) has a solution in the form of vector functions which holomorphic at the origin. We denote two sets of germs of analytic sets at zero:

$$ci(S_P) = \{z = ci(z(x)) : P_j(z(x), x) = 0; j = \ 1, 2, \dots, n\},$$

$$S_{ci(P)} = \{z = z(x) : ci(P_j(z(x), x)) = 0; j = \ 1, 2, \dots, n\}.$$

Based on the above example and these notation, one can formulate Theorem 2.1 as follows.

Theorem 3.1. *The inclusion is valid:* $ci(S_P) \subseteq S_{ci(P)}$.

Furthermore, Theorem 2.1 is equivalent to the following theorem.

Theorem 3.2. *If the commutative system of equations (4) is inconsistent, then the non-commutative system (1) is inconsistent.*

Thus, conditions of inconsistence of commutative system of equations are of interest. Naturally, the conditions for the uniqueness of a solution and the existence of infinite number of solutions of the system (1) are also of interest.

We give the following definition. Let us say that the system of equations (1) has infinitely many solutions, if the set of its solutions depends on at least one arbitrary FPS from the variables $x_1, \dots, x_n$.

So, the system of two identical equations $x_1 z_1 - z_2 x_2 = 0$ has the solution $z_1 = s x_2$, $z_2 = x_1 s$, where s is an arbitrary FPS, and therefore this system has infinitely many solutions.

Let

$$\frac{D(ci(P_1), \dots, ci(P_n))}{D(z_1, \dots, z_n)} = \det\left(\frac{\partial(ci(P_i(z, x)))}{\partial z_j}\right)$$

be the Jacobian of the system of equations (4) with respect to the variables $z_1, \dots, z_n$.

The following theorem is known.

Theorem* ([10], p. 39)**.** *Let the equality*

$$\frac{D(ci(P_1), \dots, ci(P_n))}{D(z_1, \dots, z_n)} \equiv 0$$

be fulfilled, then the system of equations (4) whenever either x has no solution in the space $\mathbb{C}^n_z$, or all its solutions in this space are nonisolated.

However, for non-commutative systems of equations (1) such an alternative (no solutions — infinitely many solutions) is not the case.

As an example, let us consider the system consisting of two identical equations:

$$x_1 z_1 + x_2 z_2 - x_1 x_2 - x_2 x_1 = 0.$$

The commutative image of this system has the Jacobian which is identically equal to zero, however, the initial non-commutative system of equations has an unique solution. Indeed, writing the equation in the form $x_1(z_1 - x_2) + x_2(z_2 - x_1) = 0$, we get that the first term belongs to the left ideal, generated by x_1 and the second one belongs to the left ideal, generated by x_2. It is obvious, that the sum of these summands can become zero only in the case, when both summands equal zero: $z_1 - x_2 = 0$, $z_2 - x_1 = 0$. Hence, the initial system of equations has the unique solution $z_1 = x_2$, $z_2 = x_1$.

The example of the system of two identical equations

$$x_1(z_1 - x_1)(z_1 - x_2) + x_2(z_2 - x_1)(z_2 - x_2) = 0$$

with four solutions shows that such systems may have any finite number of solutions.

Thus, let us note that in the case, where the Jacobian of the commutative image of the system is identically equal to zero, the initial system of non-commutative equations may: not have solutions, have a finite number of solutions, have infinitely many solutions.

Given that one equation $f = 0$ is equivalent to the system of equations $f = \ldots = f = 0$ of which the Jacobian is identically equal to zero, we will formulate the note on the solutions of these systems in relation to one non-commutative equation $P_1(z.x) = 0$: such an equation may not have solutions, have a finite and infinite number of solutions – that is the fundamental difference between equations over the field of complex numbers.

The reason for this effect is that a non-commutative polynomial equation may be equivalent to a system of such equations, for example, the equation $x_1A_1 + x_2A_2 = 0$ is equivalent, as we have seen, to the system of polynomial equations $A_1 = A_2 = 0$.

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О разрешимости систем символьных полиномиальных уравнений

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Разрабатываются подходы к решению систем некоммутативных полиномиальных уравнений в виде формальных степенных рядов (ФСР), основанные на связи с соответствующими коммутативными уравнениями. Всякому ФСР поставлен в соответствие его коммутативный образ – степенной ряд, который получается в предположении, что все символы алфавита обозначают коммутативные переменные, принимающие значения из поля комплексных чисел. Доказано, что если исходная некоммутативная система полиномиальных уравнений совместна, то и система уравнений, являющаяся ее коммутативным образом, совместна. Обратное, вообще говоря, неверно.

Показано, что в случае некоммутативного кольца система уравнений может не иметь решения, иметь конечное число решений, а также иметь бесконечно много решений, что принципиально отличается от случая комплексных переменных.

Ключевые слова: некоммутативное кольцо, полиномиальные уравнения, формальный степенной ряд, коммутативный образ.