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Bidiagonal Ranks of Completely (0-)simple Semigroups

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A bidiogonal act over a semigroup is a two-sided act, where the semigroup acts on its Cartesian power. A bidiagonal rank of a semigroup is the least power of a generating set of the bidiagonal act over this semigroup. In this paper we compute bidiagonal ranks of completely (0-)simple semigroups.

Keywords: act over a semigroup, diagonal rank, completely (0-)simple semigroup.
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A right act over a semigroup S is a set X with a map $X \times S \to X$, $(x, s) \mapsto xs$ satisfying $(xs)s' = x(ss')$ for all $x \in X$, $s, s' \in S$ (see [1]). *A left S-act Y over the semigroup S* is defined analogously: $S \times Y \to Y$, $(s, y) \mapsto sy$, $s(s'y) = (ss')y$ for all $s, s' \in S$, $y \in Y$. Let S, T be semigroups. A set Z is called an *(S, T)-act (bi-act over S and T)*, if it is a left S-act and a right T-act at the same time and $(sz)t = s(zt)$ for all $z \in Z$, $s \in S$, $t \in T$. The right S-act X, left S-act Y and (S, T)-bi-act Z may be denoted by X_S, ${}_SY$, and ${}_SZ_T$.

A generating set G of the act $(S \times S)_S$ is called *irreducible* if none of its subsets $G' \subset G$ is a generating set of this act. Clearly, any finite generating set may be reduced to an irreducible one. A generating set is called *minimal* if it is minimal with respect to power.

Note that a diagonal act over a semigroup is a unary algebra. Indeed, if S is a semigroup, then multiplication by $s \in S$ may be thought as applying unary operation $\varphi_s : x \mapsto xs$, where $x \in S$. Therefore, the following theorem is applicable to diagonal acts.

Theorem 1 (Kartashov, [2], Theorem 1)**.** *Let A be an algebra with signature $\Sigma = \{\varphi_i \mid i \in I\}$, where all operations φ_i are unary. If A is finitely generated, then any irreducible generating set of A is minimal.*

Let S be a semigroup. *A right diagonal rank* of S (*denoted by* $\operatorname{rdr} S$) is called the least power of generating sets of the diagonal rigth act of S, or

$$\operatorname{rdr} S = \min\left\{|A| \mid A \subseteq S \times S \,\wedge\, AS^1 = S \times S\right\}.$$

A bidiagonal rank $\operatorname{bdr} S$ of S is defined in a similar way.

Diagonal acts were used in [3, 4] to study conditions of wreath products to be finitely generated. Diagonal acts themselves became a subject of study in [5–7] and others. In these papers the prime problem were conditions of finite generateness of infinite diagonal acts. The notion of diagonal rank was explicitly formulated in the paper [8]. In the paper [9] one-sided diagonal ranks of completely (0-)simple semigroups were calculated. In this paper we continue the study of diagonal ranks of semigroups and calculate bidiagonal ranks of completely (0-)simple semigroups. For completeness we cite the prime results of [9] concerned with completely (0-)simple

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semigroups. These theorems may be proven using reasoning similar to the one in the presented theorems.

Theorem 2 ([9], Theorem 2). *Let S be a Rees matrix semigroup with sandwich-matrix P: $S = \mathcal{M}(G, I, \Lambda, P)$. Then the right diagonal rank of S equals $|I|^2 G$, if Λ is singleton and $|I|^2|G|^2\Lambda(\Lambda - 1)$ otherwise.*

Theorem 3 ([9], Theorem 3). *Let S be a Rees matrix semigroup with zero: $S = \mathcal{M}^0(G, I, \Lambda, P)$. Let $|I| = k$, $|G| = t$, $|\Lambda| = l$. If $l \geqslant 2$ then:*

- *if there are no zeros in P, then* $\operatorname{rdr} S = k^2t^2\left(l^2 - l\right) + 2k$;
- *if there are zeros in P, but there is no column with two or more nonzero elements, then* $\operatorname{rdr} S = k^2t^2\left(l^2 - l\right) + k^2t$;
- *otherwise,* $\operatorname{rdr} S = k^2t^2\left(l^2 - l\right)$.

If $l = 1$, then $\operatorname{rdr} S = k^2t + 2k$.

The following theorems are the body of this paper. As in the case of one-sided ranks, bidiagonal ranks of completely (0-)simple semigroups depend on sandwich-matrices insignificantly.

Theorem 4. *Let S be a completely simple semigroup: $S = \mathcal{M}(G, I, \Lambda, P)$, where G is a group of t elements and f conjugacy classes, I and Λ are index sets of k and l elements correspondingly and P is a sandwich-matrix. Then*

- *if $k, l \neq 1$, then* $\operatorname{bdr} S = t^2l(l-1)k(k-1)$;
- *if $k = 1, l \neq 1$, then* $\operatorname{bdr} S = tl(l-1)$;
- *if $l = 1, k \neq 1$, then* $\operatorname{bdr} S = tk(k-1)$;
- *if $l = 1, k = 1$, then* $\operatorname{bdr} S = f$.

Proof. Let $k, l > 1$, $M = \left\{\left((x)_{i_1\lambda_1}, (y)_{i_2\lambda_2}\right) \mid i_1 \neq i_2, \lambda_1 \neq \lambda_2\right\}$. We prove that M is an irreducible generating set of ${}_S(S \times S)_S$. Indeed, from

$$(a)_{i\lambda}\left((x)_{i_1\lambda_1}, (y)_{i_2\lambda_2}\right) = \left((ap_{\lambda i_1}x)_{i\lambda_1}, (ap_{\lambda i_2}y)_{i\lambda_2}\right),$$

and

$$\left((x)_{i_1\lambda_1}, (y)_{i_2\lambda_2}\right)(b)_{i\lambda} = \left((xp_{\lambda_1 i}b)_{i_1\lambda}, (yp_{\lambda_2 i}b)_{i_2\lambda}\right),$$

we see that pairs from M cannot be obtained from any other pairs using one-sided or two-sided multiplication. Therefore, M is a subset of any generating set.

Now we prove that M itself is a generating set. A pair of a kind $\left((u)_{j_1\mu}, (v)_{j_2\mu}\right)$, where $j_1 \neq j_2$, may be obtained from M in the following way. Take $\lambda_1 \neq \lambda_2 \in \Lambda$ and any $i \in I$. Then

$$\left((1)_{j_1\lambda_1}, \left(vu^{-1}p_{\lambda_1 i}p_{\lambda_2 i}^{-1}\right)_{j_2\lambda_2}\right)\left(p_{\lambda_1 i}^{-1}u\right)_{i\mu} = \left((u)_{j_1\mu,}, (v)_{j_2\mu}\right).$$

In a similar way we may get pairs of a kind $\left((u)_{i\lambda_1}, (v)_{i\lambda_2}\right)$, where $\lambda_1 \neq \lambda_2$.

Pairs of a kind $\left((u)_{j\mu}, (v)_{j\mu}\right)$ are obtained by multiplying the pair

$$\left(\left(p_{\lambda_1 i_2}^{-1}up_{\lambda_2 i_4}^{-1}\right)_{i_2\lambda_2}, \left(p_{\lambda_1 i_3}^{-1}vp_{\lambda_3 i_4}^{-1}\right)_{i_3\lambda_3}\right)$$

by elements $(1)_{j\lambda_1}$ and $(1)_{i_4\mu}$. Here $i_2 \neq i_3$, and $\lambda_2 \neq \lambda_3$. These pairs are in M and M is a generating set. Hence, $\operatorname{bdr} S = |M| = t^2l(l-1)k(k-1)$.

Let $l = 1$, $k > 1$. Matrix P is a vector. Using Lemma 3.6 from [10] and the remark after we can say that P consists of 1's. Let $M = \left\{ \left((1)_i , (t)_j \right) \mid i \neq j \right\}$.

Take a pair $x = \left((a)_i , (b)_j \right)$ such that $i \neq j$. Since

$$\left((1)_i , \left(ba^{-1}\right)_j \right) (a)_i = \left((a)_i , (b)_j \right),$$

then $x \in MS$. Pairs $((a)_i , (b)_i)$ we get via

$$(1)_i \left((1)_i , \left(ba^{-1}\right)_j \right) (a)_i = ((a)_i , (b)_i)\,.$$

Hence $S \times S \subseteq S^1 MS$ and M is a generating set. Now we prove that M is irreducible. It is sufficient to show that no pairs from M are products of an another pair and elements of S^1. Let $\left((1)_i , (a)_j \right) = s' \left((1)_p , (b)_q \right) s$ for some $s, s' \in S^1$, where $i \neq j$, $p \neq q$. Let $s = (c)_r$. Then $\left((1)_i , (a)_j \right) = \left((1)_p , (b)_q \right) (c)_r = \left((c)_p , (bc)_q \right)$. Hence $p = i$, $q = j$, $c = 1$ and $\left((1)_p , (b)_q \right) = \left((1)_i , (a)_j \right)$. It means that M is irreducible, so $\operatorname{bdr} S = |M| = tk(k-1)$.

Let $k = 1$, $l \neq 1$. In a similar way one can prove that $\operatorname{bdr} S = tl(l-1)$.

Let $k = 1$, $l = 1$. Now $S \simeq G$. Using Lemma 4.1 from [5] we get $\operatorname{bdr} G = f$. So $\operatorname{bdr} S = f$. □

Theorem 5. *Let S be a completely (0-)simple semigroup: $S = \mathcal{M}^0 (G, I, \Lambda, P)$, where the group G consists of t elements and f conjugacy classes, I and Λ are index sets of k and l and elements correspondingly and P is a sandwich matrix*

- *Let $k, l > 1$. If a row or a column of P has two or more non-zero elements we say that it is good. Consider the following cases.*
  1. *There are no zeros in P. Then* $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + 2$.
  2. *There are zeros in P. Moreover, there is a good row and a good column in P. Then* $\operatorname{bdr} P = t^2 k(k-1)l(l-1)$.
  3. *There is a good row in P, but no good column. Then* $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + tk(k-1)$.
  4. *There is a good column in P, but no good row. Then* $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + tl(l-1)$.
  5. *There are no good rows or columns in P. Then* $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + tk(k-1) + tl(l-1) + f$.
- *If $k = 1$, $l > 1$, then* $\operatorname{bdr} S = tl(l-1) + 2$.
- *If $l = 1$, $l > 1$ then* $\operatorname{bdr} S = tk(k-1) + 2$.
- *If $k = l = 1$, then* $\operatorname{bdr} S = f + 2$.

Proof. Throughout the following text consider the indices denoted by different symbols to be different.

Let $k, l > 1$. Action in the biact ${}_S (S \times S)_S$ is defined by the following expressions:

$$(a)_{i\lambda} \left((x)_{i_1\lambda_1} , (y)_{i_2\lambda_2} \right) = \left((ap_{\lambda i_1} x)_{i\lambda_1} , (ap_{\lambda i_2} y)_{i\lambda_2} \right),$$
$$\left((x)_{i_1\lambda_1} , (y)_{i_2\lambda_2} \right) (b)_{i\lambda} = \left((xp_{\lambda_1 i} b)_{i_1\lambda} , (yp_{\lambda_2 i} b)_{i_2\lambda} \right).$$

Divide all pairs from $S \times S$ into the following classes:

1. $\left((u)_{j_1\mu_1} , (v)_{j_2\mu_2} \right)$; 2. $\left((u)_{j\mu_1} , (v)_{j\mu_2} \right)$; 3. $\left((u)_{j_1\mu} , (v)_{j_2\mu} \right)$;
4. $\left((u)_{j\mu} , (v)_{j\mu} \right)$; 5. $\left((u)_{j\mu} , 0 \right)$; 6. $\left(0, (v)_{j\mu} \right)$; 7. $(0, 0)$.

Pairs of the class 1 do not belong to $A = S^1 \cdot (S \times S) \cdot S^1$. So we add them to the generating set. There are $t^2 k(k-1)l(l-1)$ such pairs.

Consider the class 2. Depending on P we have two cases.

- There is a good row in P. Then pairs of the class 2 are obtainable via left multiplication of the class 1 pairs.
- There are no good rows in P. Then the class 2 pairs are not obtainable from the class 1 pairs. To fix that we add to the generating set the pairs of a kind $\left((1)_{q\lambda_1}, (y)_{q\lambda_2}\right)$, where index q is arbitrary, but fixed. There are $tl(l-1)$ such pairs.

Consider pairs of the class 3. Depending of P we have the following cases.

- There is a good column in P. Pairs of the class 3 are obtainable from the class 1 pairs via right multiplication.
- There is no good columns in P. Then the class 2 pairs are not obtainable from the class 1 pairs. To fix that we add to the generating set the pairs of a kind $\left((1)_{i_1\mu}, (y)_{i_2\mu}\right)$, where index μ is arbitrary, but fixed. There are $tk(k-1)$ such pairs.

Consider pairs of the class 4. Depending on P we have the following cases.

- There is a good row and a good column in P. Then pairs of the class 4 are obtainable via two-sided action on the class 1 pairs.
- There is no good row in P, but there is a good column. Then the pairs of the class 4 are obtainable via right-sided action on the class 2 pairs. These pairs we obtain from the generating set.
- There is no good column in P, but there is a good row. Then the pairs of the class 4 are obtainable via the left-sided action on the class 3 pairs. These pairs we obtain from the generating set.
- There are no good rows or columns in P. Fix indices q, μ, pick one representative g_r from every conjugacy class of G and add to the generating set f pairs $\left((1)_{q\mu}, (g_r)_{q\mu}\right)$, $1 \leqslant r \leqslant f$. We show how to get an arbitrary pair $\left((u)_{j\nu}, (v)_{j\nu}\right)$ from these pairs. Choose $h, a, b \in G$ such that $h^{-1} g_r h = vu^{-1}$, $a = h^{-1} p_{\lambda q}^{-1}$, $b = p_{\mu i}^{-1} hu$. Then

$$(a)_{j\lambda} \left((1)_{q\mu}, (g_r)_{q\mu}\right) (b)_{i\nu} = \left((u)_{j\nu}, (v)_{j\nu}\right).$$

Consider class 5 and class 6 pairs. If there are no zeroes in P we have to add two pairs to the generating set: $\left((x)_{j\mu}, 0\right)$ и $\left(0, (x)_{j\mu}\right)$.

The pairs $(0,0)$ are obtainable from any pair.

So we have the following cases.

1. There are no zeroes in P. Then $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + 2$.
2. There is a good column and a good row in P. Then $\operatorname{bdr} P = t^2 k(k-1)l(l-1)$.
3. There is a good row in P, but no good column. Then $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + tk(k-1)$.
4. There is a good column in P, but no good row. Then $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + tl(l-1)$.
5. There are no good rows or columns in P. Then $\operatorname{bdr} P = t^2 k(k-1)l(l-1) + tk(k-1) + tl(l-1) + f$.

Let $l = 1$, $k > 1$. Then S is a left group with externally adjoined zero: $S = A \cup \{0\} = \{(g)_i \mid g \in G, i \in I\} \cup \{0\}$, where $|A| = k > 1$. By Theorem 4 the act ${}_A(A \times A)_A$ is generated by the pairs $\left((1)_i, (g)_j\right)$, where $g \in G$, $i, j \in I$. To get an irreducible generating set of the act ${}_S(S \times S)_S$ we add the pairs $\left(0, (1)_{i_0}\right)$ and $\left((1)_{i_0}, 0\right)$. Hence $\operatorname{bdr} S = tk(k-1) + 2$.

Analogously for the case $k = 1$, $l > 1$ we get $\operatorname{bdr} S = tl(l-1) + 2$.

Let $k = l = 1$. Now we need only f pairs $(1, g_r)$, $1 \leqslant r \leqslant f$ to get pairs (u, v) and two pairs $(1, 0)$ and $(0, 1)$ to get pairs $(u, 0)$ and $(0, u)$. □

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Бидиагональные ранги вполне (0-)простых полугрупп

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Диагональным биполигоном над полугруппой называется полигон, в котором полугруппа действует с двух сторон на свою декартову степень. Бидиагональным рангом полугруппы называется наименьшая мощность порождающего множества ее бидиагонального полигона. В данной работы мы вычисляем бидиагональные ранги вполне (0-)простых полугрупп.

Ключевые слова: полигон над полугруппой, диагональный ранг, вполне (0-)простая полугруппа.