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On the Correctness of Polynomial Difference Operators

Marina S. Rogozina*
Krasnoyarsk State Medical University
Partizana Zheleznyaka, 1, Krasnoyarsk, 660022
Russia

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The correctness Cauchy problem is explored for a polynomial difference operator. The easily verifiable sufficient condition correctness for the Cauchy problem for a polynomial difference operator with constant coefficients is proved whose characteristic polynomial is homogeneous.

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Difference equations arise in various areas of mathematics. Together with the method of generating functions, they yield a powerful machinery for studying enumeration problems in combinatorial analysis (see, e.g. [1]). Another source of difference equations is the discretization of differential equations. For instance, the discretization of Cauchy-Riemann equations led to the appearance of the theory of discrete analytic functions (see, e.g. [2, 3]) which found applications in the theory of Reimannian surfaces and combinatorial analysis (see, e.g. [4, 5]). The methods of discretization of a differential problem comprise an important part of the theory of difference schemes, and they also lead to difference equations (see, e.g. [6]). The theory of difference schemes is exploring ways of constructing difference schemes, explores the challenges posed difference and the convergence of the solution of the difference problem to the solution of the original differential problem, engaged justification of algorithms for solving problems of difference. An important place among these are correct.

We define the shift operators δ_j with respect to the variables x_j: $\delta_j f(x) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n)$, and polynomial difference operator of the form

$$P(\delta) = \sum_{|v| \leqslant m} c_v \delta^v,$$

where m is an order polynomial difference operator and c_v are coefficients of the difference operator, $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, $v = (v_1, v_2, \ldots, v_n)$ and $\delta^v = \delta_1^{v_1} \delta_2^{v_2} \ldots \delta_n^{v_n}$.

Let us consider the difference equation of the form

$$P(\delta) f(x) = g(x), \;\; x \in X, \tag{1}$$

where $f(x)$ is unknown function and $g(x)$ is a function defined on some fixed set $X \subset \mathbb{Z}^n$. Let us select a subset $X_0 \subset X$ of the «initial» («boundary») points from the set X. Let us formulate the problem.

Find a function $f(x)$ that satisfies the equation (1) and coincides with the given function on set X_0

$$f(x) = \varphi(x), \;\; x \in X_0. \tag{2}$$

*rogozina.marina@mail.ru

The problem (1)–(2) is called the Cauchy problem for a polynomial difference operator $P(\delta)$.

Let us denote $\|f\| = \sup_x |f(x)|$ for a function $f : X \to \mathbb{C}$.

It is said (see, e.g. [7]) that the problem of the form *(1)–(2) for a polynomial difference operator $P(\delta)$ posed, if the following conditions are satisfied:*

a) *the problem is uniquely solvable for any initial data $\varphi(x)$ and right-hand sides $g(x)$;*

b) *there are constants $M_1 > 0$, $M_2 > 0$ such that for any $g(x)$ and $\varphi(x)$ the estimate*

$$\|f(x)\| \leqslant M_1 \|g(x)\| + M_2 \|\varphi(x)\| . \tag{3}$$

Note that if the condition b) the difference operator $P(\delta)$ it said to be stable.

Thus, the difference problem (1)–(2) posed, if for any φ and g It has a unique solution and is stable.

Stability problems of the form (1)–(2) in the case of a single variable is studied in the framework of the theory of discrete dynamical systems and digital recursive filters (see, e.g. [8, 9]). Various options for determining the stability of homogeneous linear differential equations with constant coefficients mean *that all the roots of the characteristic equation do not exceed one, and if the root of the module is equal to one, it is simple. For the inhomogeneous equation stability criterion is that the roots of the module is less than one.*

In the multidimensional case, the existence and uniqueness of the solution of the problem depends on all objects involved in its formulation: the difference operator $P(\delta)$, sets X and X_0. Here are some typical situations.

The first of these, appearing usually in combinatorial analysis $X = \mathbb{Z}^n_+$ And the choice of the set on which the initial data X_0 depends on the properties of the characteristic polynomial P.

In the second case $X = \{x \in \mathbb{Z}^n, x_n \geqslant 0\}$ and as the set $X_0 \subset X$ take $X_0 = \{x \in X : x_n = 0, 1, \ldots, m-1\}$ and the characteristic polynomial is a monomial of degree m older one of the variables. In [10] to study the stability of multilayer homogeneous difference schemes of this type of theory is used amoebas of algebraic hypersurfaces. The notion of an amoeba allows us to formulate a multi-dimensional analogue of the condition that all the roots of the characteristic polynomial lie in the unit circle, i.e. stability conditions for multidimensional difference schemes.

In this paper we consider a polynomial difference operator of order m

$$P(\delta) = \sum_{|v|=m} c_v \delta^\alpha, \tag{4}$$

where $\delta^v = \delta_1^{v_1} \ldots \delta_n^{v_n}$ for a multi-index $v = (v_1, \ldots, v_n)$, with$|v| = v_1 + \ldots + v_n$, and c_v are some coefficients on the difference operator. Consider the difference equations of the form

$$P(\delta)f(x) = g(x), \;\; x \in \mathbb{Z}^n_+, \tag{5}$$

where $f(x)$ is unknown, while the function $g(x)$ on $\mathbb{Z}^n_+ = \mathbb{Z}_+ \times ... \times \mathbb{Z}_+$ is known, here $\mathbb{Z}_+$ stands for the set of nonnegative integers.

Given two points x and y of the integer lattice $\mathbb{Z}^n$ the inequality $x \geqslant y$ means that $x_i \geqslant y_i$ for $i = 1, ..., n$, while $x \ngeqslant y$ means that there is $i_0 \in \{1, ..., n\}$ with $x_{i_0} < y_{i_0}$.

Let us fix a multi-index β such that

$$|\beta| = m \;\; \text{и} \;\; c_\beta \neq 0. \tag{$*$}$$

Put $X_{0,\beta} = \{x \in \mathbb{Z}^n_+ : x \ngeqslant \beta\}$ and formulate the problem.

Find a function $f(x)$ that satisfies the equation (5) and coincides with the given function $\varphi(x)$, for $x \in X_{0,\beta}$, i.e.

$$f(x) = \varphi(x), \quad x \in X_{0,\beta}. \tag{6}$$

If $\beta \neq (m, ..., 0)$ and $\beta \neq (0, ..., m)$ then conditions (6) is called the type conditions Riquier. It is true (see [11]) easily verifiable sufficient condition for solvability of the problem (5)–(6).

If the coefficients of the homogeneous component of highest degree satisfy

$$|c_\beta| > \sum_{v\neq\beta, |v|=m} |c_v|, \tag{7}$$

then the problem (5)–(6) has the unique solution.

Condition (7) ensures the solvability of the Cauchy problem with initial data type (6). However, it generally should not be resistance problem.

Example. For a polynomial difference operator $P(\delta_1, \delta_2) = \delta_1\delta_2 - 2$ let us consider the difference equation of the form $(\delta_1\delta_2 - 2)f(x, y) = 0$ with initial data $f(0, 0) = 1$, $f(0, y) = 0$, $y > 0$, $f(x, 0) = 0$, $x > 0$. The solution to this problem is a function of $f = \begin{cases} 2^k, \ x = y = k, & k \in \mathbb{Z}^n_+, \\ 0, \ x \neq y, \end{cases}$ that is obviously unbounded, so the problem is unstable.

However, we have shown that if the *characteristic polynomial* $P(z) = \sum\limits_{v=m} c_v z^v$ *is homogeneous, then the condition (7) should also stability.*

In the case of the solvability of the problem (5)–(6) fundamental solution plays an important role, as it can be expressed through any decision (see, e.g. [10, 12, 13]).

Definition 1. *The fundamental solution of the problem (5)–(6) is a solution of $\mathcal{P}_\beta(x)$ to the equation $P(\delta)\mathcal{P}_\beta(x) = \delta_0(x)$ satisfying the initial conditions $\mathcal{P}_\beta(x) = 0$ for $x \in X_{0,\beta}$ where* $\delta_0(x) = \begin{cases} 0, \ if\, x \neq 0, \\ 1, \ if\, x = 0. \end{cases}$

First, we note that *in the case where the characteristic polynomial $P(z)$ is homogeneous* the fundamental solution has the form $\mathcal{P}_\beta(x) = \dfrac{1}{c_\beta}\delta_0(x - \beta)$.

Indeed, for $x \geqslant 0$ we have

$$P(\delta)\frac{1}{c_\beta}\delta_0(x-\beta) = \frac{1}{c_\beta}\sum_{|v|=m} c_v\delta^v\delta_0(x-\beta) = \sum_{|\nu|=m}\frac{c_v}{c_\beta}\delta_0(x+v-\beta).$$

If $v \neq \beta$ then at least one of the differences $v_j - \beta_j > 0$ and $x_j \geqslant 0$ so $\delta_0(x + v - \beta) = 0$ and hence

$$P(\delta)\frac{1}{c_\beta}\delta_0(x-\beta) = \frac{c_\beta}{c_\beta}\delta_0(x) = \delta_0(x).$$

Next to $x \in X_{0,\beta}$ we have $x \geqslant 0$, $x \ngeqslant \beta$ therefore $x - \beta \ngeqslant 0$, i.e. $x_j - \beta_j < 0$ for some j j, then $\mathcal{P}_\beta(x) = \dfrac{1}{c_\beta}\delta_0(x-\beta) = 0$.

The main result of this work.

Theorem. *If a polynomial difference operator of the form (4) and the condition (7), the problem (5)–(6) is correct.*

Proof. We use the following formula (see. [13])

$$f(x) = \sum_{k\geqslant 0, k\ngeqslant\beta} \varphi(k) \sum_{\nu\nleqslant k} c_\nu \mathcal{P}_\beta\left(x+\nu-k\right), \tag{8}$$

to solve the problem (5)–(6).

Let us show that in the formula (8) the number of terms is finite. By the definition of fundamental solution,$\mathcal{P}_\beta(x) = 0$ for $x \ngeqslant \beta$. Therefore, in the expression for $f(x)$ we sum over k and ν with $x+\nu-k \geqslant \beta$, i.e. $k \leqslant x+\nu-\beta$. For fixed x and β the system of inequalities $k \geqslant 0$, $k \leqslant x+\nu-\beta$ and $0 \leqslant |v| \leqslant m$ determines a bounded set in $\mathbb{R}^n_k \times \mathbb{R}^n_\nu$.

Verify that $f(x)$ defined in (8) satisfies (5). Indeed, for $x \geqslant 0$ we have

$$P(\delta) f(x) = \sum_{k\geqslant 0,\, k\ngeqslant\beta} \varphi(k) \sum_{\nu\nleqslant k} c_\nu \delta_0(x+\nu-k).$$

Since $\nu \nleqslant k$, there is j with $\nu_j - y_j > 0$, hence $x_j + \nu_j - y_j > 0$ and then $\delta_0(x+\nu-y) = 0$, Therefore, $P(\delta)f(x) = 0$.

It remains to verify $f(x)$ in (8) satisfies (6).

For $\nu \leqslant k$ we have $x+\nu-k \leqslant x$, and since $x \ngeqslant \beta$, it follows that $\mathcal{P}_\beta(x+\nu-k) = 0$. Then we can express $f(x)$ as

$$f(x) = \sum_{k\geqslant 0,\, k\ngeqslant\beta} \varphi(k) \sum_{0\leqslant|\nu|\leqslant m} c_\nu \mathcal{P}_\beta(x+\nu-k) = \sum_{k\geqslant 0,\, k\ngeqslant\beta} \varphi(k)\delta_0(x-k) = \varphi(k).$$

Thus, in the formula (8) for solutions $f(x)$ problem (5)–(6) the number of terms is finite but generally speaking, depends on the x. Let us show, that in the case where the characteristic polynomial $P(z)$ difference equation (5) is homogeneous, the number of terms will not depend on the x.

Let us fix $x \geqslant 0$ in the formula (8) are the sum of multi-index k and ν such that

$$k \geqslant 0, \;\; k \ngeqslant \beta, \;\; \nu \nleqslant k, \;\; \nu \geqslant 0. \tag{**}$$

and $k - \nu = x - \beta$.

Since the characteristic polynomial $P(z)$ is homogeneous of degree m m then for all j the inequality $\nu_j \leqslant m$ and by (**) the inequality $k_j \geqslant 0$. On the plane $\mathbb{R}^2_{\nu_j,k_j}$ of variable ν_j, k_j let us consider the equation of the line $k_j - \nu_j = const$. The number of lattice points in the intersection of this line with the plane $\{(\nu_j, k_j) : 0 \leqslant \nu_j \leqslant m\}$ equal to $(m+1)$ for any value of the constant. It follows that for any x the number of solutions of equations $k_j - \nu_j = x_j - \beta_j$ satisfying (**) are less than $(m+1)$. A number of solutions ν, k system of equations $k - \nu = x - \beta$ are not more than $(m+1)^n$, i.e. it does not depend on the x.

Let us prove the stability of the problem (5)–(6). We use the formula (8) for the solution of the problem (5)–(6).

$$|f(x)| \leqslant \left| \sum_{k\geqslant 0,\, k\ngeqslant\beta} \varphi(k) \sum_{\nu\nleqslant k} c_\nu \mathcal{P}_\beta(x+\nu-k) \right| \leqslant \sum_{k\geqslant 0,\, k\ngeqslant\beta} |\varphi(k)| \sum_{\nu\nleqslant k} \left| c_\nu \mathcal{P}_\beta(x+\nu-k) \right|.$$

Since the characteristic polynomial is homogeneous, the fundamental solution has the form $\mathcal{P}_\beta(x) = \dfrac{1}{c_\beta}\delta_0\left(x-\beta\right)$. We find that

$$|f(x)| \leqslant \sum_{k\geqslant 0,\, k\ngeqslant\beta} |\varphi(k)| \sum_{\nu\nleqslant k} \frac{|c_\nu|}{|c_\beta|} \delta(x+\nu-k) \leqslant C \cdot (m+1)^n .$$

This means stability of the problem (5)–(6). □

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О корректности полиномиальных разностных операторов

Марина С. Рогозина

Исследуется корректность задачи Коши для полиномиального разностного оператора. Доказано легко проверяемое достаточное условие корректности задачи Коши для полиномиального разностного оператора с постоянными коэффициентами, характеристический многочлен которого является однородным.

Ключевые слова: полиномиальный разностный оператор, задача Коши, корректность.