Journal of Siberian Federal University. Mathematics & Physics 2015, 8(4), 416–425

УДК 517.9

On Solvability of one Class of Nonlinear Integral-differential Equation with Hammerstein Non-compact Operator Arising in a Theory of Income Distribution

Aghavard Kh. Khachatryan
Khachatur A. Khachatryan*
Tigran H. Sardaryan
Institute of Mathematics of NAS RA
Marshal Baghramyan, 24/5, Yerevan, 0009
Armenia

Received 06.08.2015, received in revised form 03.09.2015, accepted 05.10.2015

In present paper we investigate a class of nonlinear integral- differential equation with Hammerstein noncompact operator which has direct application in a theory of income distribution. We prove solvability of the class of equations in special weighted Sobolev space. The results of numerical calculations are also presented.

Keywords: Hammerstein operator, weighted Sobolev space, monotony, iteration, Caratheodory's condition.
DOI: 10.17516/1997-1397-2015-8-4-416-425

Introduction

In this paper we study the following initial problem for the nonlinear integral-differential equation with noncompact Hammerstein operator

$$\frac{df}{dx} + \lambda_0 (x, f(x)) = \int_0^\infty K(x-t)\, \lambda_1 (t, f(t))\, dt, \quad x \in \mathbb{R}^+ \equiv [0, +\infty), \tag{1}$$

$$f(0) = 0 \tag{2}$$

where $f(x)$ is a real function defined on $\mathbb{R}^+$.

The functions $\{\lambda_j(x,u)\}_{j=0,1}$ are defined on set $\mathbb{R}^+ \times \mathbb{R}$. They take real values and satisfy condition of criticality:

$$\lambda_j(x,0) \equiv 0, \quad \forall x \in \mathbb{R}^+, \quad j = 0,1. \tag{3}$$

The kernel $K(x)$ admits the following representation:

$$K(x) = \int_a^b e^{-|x|s} G(s)\, ds, \quad x \in \mathbb{R} \equiv (-\infty, +\infty), \tag{4}$$

where

$$G \in C[a,\ b), \quad G(s) > 0, \quad s \in [a,\ b), \quad 0 < a < b \leqslant +\infty, \tag{5}$$

moreover

$$\mu = 2\int_a^b \frac{G(s)}{s} ds < +\infty. \tag{6}$$

*Khach82@rambler.ru

We assume that there exists a number $\alpha>0$, such that

$$\lambda^{\sharp}(x,u) \equiv \alpha u - \lambda_0(x,u) \geqslant 0, \quad (x,u) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{7}$$

Problem (1)–(2) has direct application in econometrics, namely, in the theory of income distribution in one product economics (see. [1–4]). Unknown function $f(x)$ plays a role of distribution density, i.e. $f(x)\,dx$ is a number of economic agents which have incomes in the interval $(x, x+dx)$. Function λ_0 characterizes the growth of capital and savings, bankruptcy, disappearance of economic enterprises, taxes and etc. Function $\lambda_1(x,u)$ describes nonlinear dependence of distribution function. Kernel $K(x)$ is the redistribution function. It is caused by various economic factors: capital transfer, the emergence of new enterprises, association of several companies, the disappearance of old enterprises, property transfer to other economic organizations in the case of inheritance and so on. Numbers $\mu = \int_{-\infty}^{+\infty} K(x)\,dx$ and α are free parameters. They play essential role in our further consideration. The class of nonlinear equations considered in the paper is the nonlinear analogue of the class of linear equations investigated by J. D. Sargan (see [1]). For the first time above mentioned class of nonlinear equations was studied by A. Kh. Khachatryan and Kh. A. Khachatryan (see [2]). The equation in linear approximation was obtained with $\lambda_0(x,u) = c_0 u$, $\lambda_1(x,u) = c_1 u$, $c_0, c_1 = const$ and linear approximation is not real [1]. The peculiarity of corresponding nonlinear integro-differential equations and complexity of their study are the following:

1. Operators which generate corresponding equations are nonlinear.

2. Corresponding nonlinear operators are noncompact operators. Moreover, the linear minorant (or majorant) for these operators are the Wiener-Hopf type operators. It is well known that Wiener-Hopf type operators are also noncompact operators.

3. Another important peculiarity of these equations is that they admit zero solution (trivial solution). Then, it is necessary to construct a nonlinear positive solution, i.e. to clarify whether or not corresponding nonlinear operators have criticality property.

Due to these facts previously known fixed points principles (Shauder, Krasnoselskii, Browder, Brouwer's theorems) for solvability of corresponding nonlinear integral equations are not applicable.

Equation (1) was studied [5] in the case when

$$\lambda^{\sharp}(x,u) \equiv 0, \quad \lambda_1(x,u) \equiv G_0(u).$$

Here $0 \leqslant G_0(u) \leqslant u$, $u \in [0,\eta]$, $G_0 \in C[0,\eta]$, $G_0(u) \uparrow$ by u on $[0,\eta]$ for some $\eta>0$, and $f(0) = y_0$, $0 < y_0 \leqslant \eta$. This equation (1) was also studied [6] in the case when

$$\lambda^{\sharp}(x,u) \geqslant 0, \quad \lambda^{\sharp}(x,\eta) \leqslant \eta \int_x^{\infty} K(u)\,du, \quad 0 \leqslant \lambda_1(x,u) \leqslant \beta G_1(u), \quad \beta \in (0,1),$$

where $0 \leqslant G_1(u) \uparrow$ by u on $[0,\eta]$, $G_1(\eta) = \eta$ (η is the first positive root of equation $G_1(u) = u$), $G_1(0) = 0$ and G_1 satisfies Lipschitz condition on interval $[0,\eta]$, $f(0) = y_0$, $0 < y_0 \leqslant \eta$. It should be noted that equation (1) was studied in [5] and [6] when

$$\alpha \geqslant \mu. \tag{8}$$

In linear approximation ($\lambda_0(x,u) \equiv 0$, $\lambda_1(x,u) \equiv u$) this corresponds to dissipative and conservative cases. Obviously, when $\alpha \leqslant \mu$ we have conservative and supercritical cases, respectively.

Present paper is devoted to study and solution of problem (1)–(2) under very different assumptions regarding functions $\{\lambda_j(x,u)\}_{j=0,1}$ when

$$\mu \geqslant \alpha. \tag{9}$$

Solvability of problem (1)–(2) in special weighted Sobolev space is proved. The examples of functions $\{\lambda_j(x,u)\}_{j=0,1}$ are given. Specific example of the equation is considered. Algorithm of numerical solution of this equation is described and some results of numerical calculations are given.

1. Notations, auxiliary facts and formulation of basic result

In accordance with (6) condition (9) can be written as

$$æ \equiv \frac{\mu}{\alpha} = \frac{2}{\alpha}\int_a^b \frac{G(s)}{s}ds \geqslant 1. \tag{10}$$

For arbitrary number $\varepsilon \in (0, min(\alpha, a))$ we consider the function

$$T_\varepsilon(x) = e^{\varepsilon x}\int_0^\infty K(x-z)\,e^{-\alpha z}dz, \quad x \in \mathbb{R}. \tag{11}$$

From representation (4) for kernel K it follows that at $\varepsilon \in (0, \min(\alpha, a))$

$$0 \leqslant T_\varepsilon(\cdot) \in L_1(-\infty, +\infty). \tag{12}$$

On interval $\mathcal{I} \equiv (0, \min(\alpha, a))$ we introduce the function

$$\begin{gathered}\mathcal{X}(\varepsilon) = \int_{-\infty}^0 T_\varepsilon(x)\,dx = \int_0^\infty e^{-\alpha z}\int_{-\infty}^0 e^{\varepsilon x}K(x-z)\,dxdz = \int_0^\infty e^{-\alpha z}\int_{-\infty}^{-z} K(\tau)\,e^{\varepsilon(z+\tau)}d\tau dz = \\ = \int_0^\infty e^{-\alpha z}e^{\varepsilon z}\int_z^\infty K(u)e^{-\varepsilon u}dudz = \int_0^\infty e^{-(\alpha-\varepsilon)z}\int_a^b\int_z^\infty e^{-(s+\varepsilon)u}duG(s)\,dsdz = \\ = \int_a^b \frac{G(s)}{(s+\varepsilon)(s+\alpha)}ds.\end{gathered}$$

We note that

$$\text{I)} \quad \mathcal{X}(\varepsilon) \downarrow \text{by } \varepsilon \text{ on } \mathcal{I}, \tag{13}$$

$$\text{II)} \quad \mathcal{X} \in C(\mathcal{I}), \tag{14}$$

$$\text{III)} \quad \mathcal{X}(+0) = \lim_{\varepsilon \to 0+}\mathcal{X}(\varepsilon) = \int_a^b \frac{G(s)}{s(s+\alpha)}ds \equiv \gamma. \tag{15}$$

Therefore, by Cauchy theorem there exists number $\varepsilon_0 \in \mathcal{I}$ such that

$$\mathcal{X}(\varepsilon_0) > \frac{\gamma}{2}. \tag{16}$$

It is obvious that ε_0 is determined nonuniquely, i.e., if for some ε_0 inequality (16) is fulfilled, then for $\forall\varepsilon \in (0, \varepsilon_0)$

$$\mathcal{X}(\varepsilon) > \frac{\gamma}{2}. \tag{17}$$

Let us consider the set:

$$Q = \left\{\varepsilon \in \mathcal{I}\colon \mathcal{X}(\varepsilon) > \frac{\gamma}{2}\right\} \subset \mathcal{I}. \tag{18}$$

It is obvious that Q is the bounded set and hence there exists $\widetilde{\varepsilon} \equiv \sup Q < +\infty$. It follows from the structure of the set Q that

$$\mathcal{X}(\widetilde{\varepsilon}) \geqslant \frac{\gamma}{2}, \quad \widetilde{\varepsilon} \in \mathcal{I}. \tag{19}$$

Number $\widetilde{\varepsilon}$ will plays an important role in our future considerations.
We assume that functions $\{\lambda_j(x,u)\}_{j=0,1}$ satisfy the following conditions:

a) $\lambda^\sharp(x,u)\uparrow$ by u on $[\rho_{\widetilde{\varepsilon}}(x),\ +\infty)$ for each fixed $x\in\mathbb{R}^+$, where

$$\rho_{\widetilde{\varepsilon}}(x)\equiv\frac{e^{-\widetilde{\varepsilon}x}-e^{-\alpha x}}{\alpha-\widetilde{\varepsilon}},\quad x\in\mathbb{R}^+; \tag{20}$$

b) $$l\equiv\int_0^\infty \underset{u\geq 0}{essup}\,\lambda^\sharp(x,u)\,dx<+\infty; \tag{21}$$

c) $\lambda_j\in Carat_u(\mathbb{R}^+\times\mathbb{R}^+)$, i. e., functions $\{\lambda_j(x,u)\}_{j=0,1}$ satisfy Caratheodory condition by second argument (for each fixed $u\in\mathbb{R}^+$ functions $\{\lambda_j(x,u)\}_{j=0,1}$ are measurable with respect to x on set $\mathbb{R}^+$ and they are continuous with respect to u on $\mathbb{R}^+$ for almost all $x\in\mathbb{R}^+$);

d) there exist integrable on $\mathbb{R}^+$ functions $\beta_{\widetilde{\varepsilon}}(x)$ such that

$$\beta_{\widetilde{\varepsilon}}(x)\geqslant\frac{2}{\gamma}\rho_{\widetilde{\varepsilon}}(x),\ x\in\mathbb{R}^+, \tag{22}$$

$$\lambda_1(x,\rho_{\widetilde{\varepsilon}}(x))\geqslant\frac{2}{\gamma}\rho_{\widetilde{\varepsilon}}(x),\quad \lambda_1(x,u)\leqslant\frac{u}{æ}+\beta_{\widetilde{\varepsilon}}(x),\quad u\geqslant\rho_{\widetilde{\varepsilon}}(x),\quad x\in\mathbb{R}^+; \tag{23}$$

e) for each fixed $x\in\mathbb{R}^+$ function $\lambda_1(x,u)\uparrow$ by u on $[\rho_{\widetilde{\varepsilon}}(x),\ +\infty)$.

The basic result of the paper is the following:

Теорема 1. *Let us assume that kernel $K(x)$ permits representation (4). Let $\widetilde{\varepsilon}=\sup Q$, where set Q is defined by formula (18), and functions $\{\lambda_j(x,u)\}_{j=0,1}$satisfy conditions (7), (20)–(23). Then problem (1)–(2), apart from trivial solution, has also identically nonzero nonnegative solution in the following weighted Sobolev space:*

$$f\in W^1_{1,h}\left(\mathbb{R}^+\right)\equiv\left\{\varphi(x):\ \varphi^{(j)}(x)\cdot h(x)\in L_1\left(\mathbb{R}^+\right),\ j=0,1;\ h(x)\equiv\int_a^b e^{-xs}\frac{G(s)}{s(s+\alpha)}ds\right\},$$

where $\varphi^{(j)}(x)$is the j-th derivative of function $\varphi(x)$.

2. Proof

We denote

$$\psi(x)\equiv\frac{df}{dx}+\alpha f(x),\quad x\in\mathbb{R}^+. \tag{24}$$

Then taking into account condition (2), from (1) we obtain the following nonlinear integral equation:

$$\psi(x)=\lambda^\sharp\left(x,\int_0^x e^{-\alpha(x-t}\psi(t)\,dt\right)+\int_0^\infty K(x-t)\,\lambda_1\left(t,\int_0^t e^{-\alpha(t-u)}\psi(u)\,du\right)dt,\quad x\geqslant 0. \tag{25}$$

With respect to function $\psi(x)$.
We consider the following successive approximations:

$$\psi_{n+1}(x)=\lambda^\sharp\left(x,\int_0^x e^{-\alpha(x-t)}\psi_n(t)\,dt\right)+\int_0^\infty K(x-t)\,\lambda_1\left(t,\int_0^t e^{-\alpha(t-u)}\psi_n(u)\,du\right)dt,$$
$$\psi_0(x)=e^{-\widetilde{\varepsilon}x},\quad x\in\mathbb{R}^+,\quad n=0,1,2,\ldots. \tag{26}$$

It is easy to check by induction with respect to n that

$$\psi_n(x)\uparrow \text{by } \; n, \quad x\in\mathbb{R}^+. \tag{27}$$

First we proof that

$$\psi_1(x)\geqslant\psi_0(x), \;\; x\in\mathbb{R}^+. \tag{28}$$

Taking into consideration (23), from (26) we have

$$\begin{aligned}\psi_1(x)\geqslant\int_0^\infty K(x-t)\,\lambda_1\left(t,\int_0^t e^{-\alpha(t-u)}e^{-\widetilde{\varepsilon}u}du\right)dt=\int_0^\infty K(x-t)\,\lambda_1(t,\rho_{\widetilde{\varepsilon}}(t))\,dt\geqslant\\ \geqslant\frac{2}{\gamma}\int_0^\infty K(x-t)\,\rho_{\widetilde{\varepsilon}}(t)\,dt=\frac{2}{\gamma}\int_0^\infty K(x-t)\int_0^t e^{-\alpha(t-u)}e^{-\widetilde{\varepsilon}u}dudt=\\ =\frac{2}{\gamma}\int_0^\infty e^{-\widetilde{\varepsilon}u}\int_u^\infty K(x-t)\,e^{-\alpha(t-u)}dtdu=\frac{2}{\gamma}\int_0^\infty e^{-\widetilde{\varepsilon}u}\int_0^\infty K(x-u-z)\,e^{-\alpha z}dzdu=\\ =\frac{2}{\gamma}\int_0^\infty e^{-\widetilde{\varepsilon}u}T_{\widetilde{\varepsilon}}(x-u)\,e^{-\widetilde{\varepsilon}(x-u)}du=\frac{2}{\gamma}e^{-\widetilde{\varepsilon}x}\int_{-\infty}^x T_{\widetilde{\varepsilon}}(\tau)\,d\tau\geqslant\frac{2}{\gamma}e^{-\widetilde{\varepsilon}x}\int_{-\infty}^0 T_{\widetilde{\varepsilon}}(\tau)\,d\tau=\\ =\frac{2}{\gamma}e^{-\widetilde{\varepsilon}x}\mathcal{X}(\widetilde{\varepsilon})\geqslant e^{-\widetilde{\varepsilon}x}=\psi_0(x),\end{aligned}$$

because $\mathcal{X}(\widetilde{\varepsilon})\geqslant\dfrac{\gamma}{2}$ (see formula (19)).

Let $\psi_n(x)\geqslant\psi_{n-1}(x)$, $x\in\mathbb{R}^+$ for any $n\in\mathbb{N}$. Then due to monotony of functions $\lambda^\sharp(x,u)$ and $\lambda_1(x,u)$ by u on $[\rho_{\widetilde{\varepsilon}}(x),\ +\infty)$ (see conditions a) and e)), from (26) we obtain

$$\begin{aligned}\psi_{n+1}(x)\geqslant\lambda^\sharp\left(x,\int_0^x e^{-\alpha(x-t)}\psi_{n-1}(t)\,dt\right)+\\ +\int_0^\infty K(x-t)\,\lambda_1\left(t,\int_0^t e^{-\alpha(t-u)}\psi_{n-1}(u)\,du\right)dt=\psi_n(x).\end{aligned}$$

It is also easy to verify that

$$\psi_n\in L_1(\mathbb{R}^+), \quad n=0,1,2,3,\ldots. \tag{29}$$

Taking into account conditions (21), (23), (27), (29), a), e) and Fubini's theorem (see [7]), from (21) we have

$$\begin{aligned}\int_0^\infty\psi_{n+1}(x)\,dx\leqslant l+\int_0^\infty\int_0^\infty K(x-t)\,\lambda_1\left(t,\int_0^t e^{-\alpha(t-u)}\psi_{n+1}(u)\,du\right)dtdx\leqslant\\ \leqslant l+\frac{1}{æ}\int_0^\infty\int_0^\infty K(x-t)\left(\int_0^t e^{-\alpha(t-u)}\psi_{n+1}(u)\,du+æ\cdot\beta_{\widetilde{\varepsilon}}(t)\right)dtdx\leqslant\\ \leqslant l+æ\alpha\int_0^\infty\beta_{\widetilde{\varepsilon}}(t)\,dt+\frac{1}{æ}\int_0^\infty\int_0^\infty K(x-t)\int_0^t e^{-\alpha(t-u)}\psi_{n+1}(u)\,dudtdx\equiv I,\end{aligned} \tag{30}$$

as

$$\begin{aligned}æ\alpha\int_0^\infty\beta_{\widetilde{\varepsilon}}(t)\,dt=\int_0^\infty\beta_{\widetilde{\varepsilon}}(t)\int_{-\infty}^{+\infty}K(u)\,du\,dt\geqslant\int_0^\infty\beta_{\widetilde{\varepsilon}}(t)\int_{-t}^{+\infty}K(u)\,dudt=\\ =\int_0^\infty\int_0^\infty K(x-t)\,\beta_{\widetilde{\varepsilon}}(t)\,dtdx.\end{aligned}$$

Changing integration order in the last integral in (30), we obtain

$$I = l + æ\alpha \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\, dt + \frac{1}{æ}\int_0^\infty \psi_{n+1}(u) \int_0^\infty T(x-u)\, dx du, \tag{31}$$

where

$$T(\tau) \equiv \int_0^\infty K(\tau - z)\, e^{-\alpha z} dz, \quad \tau \in \mathbb{R}. \tag{32}$$

We note that

$$\int_{-\infty}^{\infty} T(\tau)\, d\tau = æ. \tag{33}$$

Therefore, from (33) and (31) we have

$$I = l + æ\alpha \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\, dt + \frac{1}{æ}\int_0^\infty \psi_{n+1}(u) \left(æ - \int_{-\infty}^{-u} T(\tau)\, d\tau \right) du. \tag{34}$$

Chain of inequalities (30) and equality (34) result in the following inequalities:

$$\int_0^\infty \psi_{n+1}(u) \int_{-\infty}^{-u} T(\tau)\, d\tau du \leqslant læ + æ^2\alpha \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\, dt. \tag{35}$$

Now we verify that

$$h(u) = \int_{-\infty}^{-u} T(\tau)\, d\tau, \quad u \geqslant 0. \tag{36}$$

Indeed, taking into consideration (4) and (32), we obtain

$$\int_{-\infty}^{-u} T(\tau)\, d\tau = \int_{-\infty}^{-u} \int_0^\infty K(\tau - z)\, e^{-\alpha z} dz d\tau = \int_u^\infty \int_0^\infty K(-y-z)\, e^{-\alpha z} dz dy =$$

$$= \int_u^\infty \int_0^\infty K(y+z)\, e^{-\alpha z} dz dy = \int_u^\infty \int_a^b \int_0^\infty e^{-s(y+z)} e^{-\alpha z} dz \cdot G(s)\, ds dy =$$

$$= \int_a^b e^{-us} \frac{G(s)}{s(s+\alpha)} ds = h(u).$$

Thus, taking into account (36) and (35), we arrive to the following inequalities:

$$\int_0^\infty \psi_{n+1}(u)\, h(u)\, du \leqslant læ + æ^2\alpha \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\, dt. \tag{37}$$

As $\psi_n(u) \uparrow$ by n, $h(u) \geqslant 0$, then from (37) we conclude that sequence of functions $\{\psi_n(u)\}_{n=0}^\infty$ has pointwise limit as $n \to +\infty$:

$$\lim_{n\to\infty} \psi_n(u) \equiv \psi(u).$$

Moreover, due to Carathedory condition (see condition c)) and B. Levi's theorem the limit function ψ satisfies equation (25). It also follows from (37) and (27) that

$$\psi(u) \geqslant \frac{2}{\gamma} \int_0^\infty K(u-t)\, \rho_{\widetilde{\varepsilon}}(t)\, dt, \quad u \geqslant 0, \tag{38}$$

$$\int_0^\infty \psi(u)\, h(u)\, du \leqslant læ + æ^2\alpha \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\, dt. \tag{39}$$

Upon solving simple Cauchy problem (24) and (2), we obtain

$$f(x) = \int_0^x e^{-\alpha(x-t)}\psi(t)\,dt. \tag{40}$$

To complete the proof one needs to verify that $f \in W^1_{1,h}(\mathbb{R}^+)$. First we prove that $f \cdot h \in L_1(\mathbb{R}^+)$. Because $h(u) \downarrow$ by u on $\mathbb{R}^+$ and, taking into account inequality (39), for arbitrary $\delta > 0$ from (40) we get

$$\int_0^\delta f(x)\,h(x)\,dx = \int_0^\delta h(x) \int_0^x e^{-\alpha(x-u)}\psi(u)\,du dx = \int_0^\delta \psi(u)\,e^{\alpha u} \int_u^\delta e^{-\alpha x} h(x)\,dx du \leqslant$$
$$\leqslant \frac{1}{\alpha}\int_0^\delta \psi(u)\,e^{\alpha u} h(u)\left(e^{-\alpha u} - e^{-\alpha\delta}\right)du \leqslant \frac{1}{\alpha}\int_0^\delta \psi(u)\,h(u)\,du \leqslant \frac{1}{\alpha}\int_0^\infty \psi(u)\,h(u)\,du \leqslant$$
$$\leqslant \frac{l æ}{\alpha} + æ^2 \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\,dt.$$

Therefore, as $\delta \to +\infty$ we obtain

$$\int_0^\infty f(x)\,h(x)\,dx \leqslant \frac{l æ}{\alpha} + æ^2 \int_0^\infty \beta_{\widetilde{\varepsilon}}(t)\,dt. \tag{41}$$

Because

$$f'(x) = \psi(x) - \alpha f(x),$$

then due to (39) and (41) we have $f' \cdot h \in L_1(\mathbb{R}^+)$.

Thus $f \in W^1_{1,h}(\mathbb{R}^+)$. Theorem is proved.

3. Example of functions $\lambda_0(x,u)$ and $\lambda_1(x,u)$

Now we present examples of functions $\{\lambda_j(x,u)\}_{j=0,\ 1}$ for which all conditions of the formulated theorem are fulfilled.

For a function $\lambda_1(x,u)$ we consider the following example:

$$\lambda_1(x,u) = \frac{2}{\gamma}\sqrt{u\rho_{\widetilde{\varepsilon}}(x)}. \tag{42}$$

If we take $\beta_{\widetilde{\varepsilon}}(x) = \frac{æ}{\gamma^2}e^{-\widetilde{\varepsilon}x}$, then all conditions will be implemented for $\lambda_1(x,u)$. Firstly, we note that

$$\lambda_1(x,0) \equiv 0, \quad \forall x \in \mathbb{R}, \quad \frac{\partial \lambda_1}{\partial u} = \frac{\sqrt{\rho_{\widetilde{\varepsilon}}(x)}}{\gamma\sqrt{u}} \geqslant 0.$$

Therefore $\lambda_1(x,u) \uparrow$ by u. Secondly, function

$$\lambda_1(x, \rho_{\widetilde{\varepsilon}}(x)) = \frac{2}{\gamma}\rho_{\widetilde{\varepsilon}}(x)$$

is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ with respect to all arguments, and hence satisfies Caratheodory condition on $\mathbb{R}^+ \times \mathbb{R}^+$ with respect to second argument.

We verify that the following inequality holds:

$$\frac{2}{\gamma}\sqrt{u\rho_{\widetilde{\varepsilon}}(x)} \leqslant \frac{u}{æ} + \beta_{\widetilde{\varepsilon}}(x), \tag{43}$$

where $\beta_{\widetilde{\varepsilon}}(x) \geqslant \dfrac{æ}{\gamma^2}\rho_{\widetilde{\varepsilon}}(x)$ and $\beta_{\widetilde{\varepsilon}} \in L_1(\mathbb{R}^+)$. Because u, $\rho_{\widetilde{\varepsilon}}$, $\beta_{\widetilde{\varepsilon}} \geqslant 0$ then the last inequality is equivalent to

$$\frac{u^2}{æ^2}+\left(\frac{2\beta_{\widetilde{\varepsilon}}(x)}{æ}-\frac{4}{\gamma^2}\rho_{\widetilde{\varepsilon}}(x)\right)u+\beta^2_{\widetilde{\varepsilon}}(x) \geqslant 0. \tag{44}$$

It is obvious that inequality (44) is true if

$$\left(\frac{2\beta_{\widetilde{\varepsilon}}(x)}{æ}-\frac{4}{\gamma^2}\rho_{\widetilde{\varepsilon}}(x)\right)^2-\frac{4\beta^2_{\widetilde{\varepsilon}}(x)}{æ^2} \leqslant 0$$

or

$$\beta_{\widetilde{\varepsilon}}(x) \geqslant \frac{æ}{\gamma^2}\rho_{\widetilde{\varepsilon}}(x). \tag{45}$$

Thus one needs to verify inequality $\beta_{\widetilde{\varepsilon}}(x)=\dfrac{æ}{\gamma^2}\rho_{\widetilde{\varepsilon}}(x) \geqslant \dfrac{2}{\gamma}\rho_{\widetilde{\varepsilon}}(x)$. This inequality is true because

$$æ=\frac{2}{\alpha}\int_a^b \frac{G(s)}{s}ds \geqslant \int_a^b \frac{2G(s)}{s(s+\alpha)}ds= 2\gamma.$$

Now we consider the example of function $\lambda_0(x,u)$:

$$\lambda_0(x,u)=\alpha u-\lambda^\sharp(x,u), \tag{46}$$

where

$$\lambda^\sharp(x,u)=\frac{u}{\sqrt{u^2x^2+u+1}}\left(e^{-mx}-e^{-qx}\right), \quad q>m>0, \quad (x,u)\in\mathbb{R}^+\times\mathbb{R}^+. \tag{47}$$

We have

$$\lambda^\sharp(x,u) \geqslant 0, \qquad \frac{\partial\lambda^\sharp(x,u)}{\partial u}=\frac{u+1}{\sqrt{u^2x^2+u+1}\cdot(u^2x^2+u+1)}>0, \tag{48}$$

hence $\lambda^\sharp(x,u)\uparrow$ by u. Since $\lambda^\sharp(x,u)$ is continuous on $\mathbb{R}^+\times\mathbb{R}^+$ with respect to all arguments then $\lambda_0 \in Carat_u(\mathbb{R}^+\times\mathbb{R}^+)$. We also note that

$$\int_0^\infty \text{supess}\lambda^\sharp(x,u)\,dx \leqslant \int_0^\infty \frac{e^{-mx}-e^{-qx}}{x}dx=\ln\frac{q}{m}<+\infty,$$

and hence $l \equiv \displaystyle\int_0^\infty \operatorname*{essup}_{u\geqslant 0}\lambda^\sharp(x,u)\,dx \leqslant \ln\frac{q}{m}<+\infty$.

4. Algorithm of numerical solution and results of numerical calculations

A brief description of algorithm of numerical calculations are given in this section.

For kernal $K(x)$ we choose integral-power function

$$K(x)=E_1(x)=\frac{1}{2}\int_1^\infty e^{-|x|s}\frac{ds}{s}, \quad G(s)=\frac{1}{2s}, \quad a=1, \quad b=\infty, \tag{49}$$

which arises in the case of generalized Paretto law (see [2–4]).

Note that in this case $\mu=1$. In this example functions λ_1 and λ_0 are

$$\lambda_1(x,u)=\frac{2}{\gamma}\sqrt{u\rho(x)},$$

$$\lambda_0(x,u)=\alpha u-\frac{u}{\sqrt{u^2x^2+u+1}}\left(e^{-mx}-e^{-qx}\right),\quad q>m>0,$$

where

$$\rho(x)=\frac{e^{-\varepsilon x}-e^{-\alpha x}}{\alpha-\varepsilon},\quad x\in\mathbb{R}^+,\quad \alpha\le\mu\equiv 1,$$

$$\gamma=\frac{1}{2}\int_1^\infty\frac{ds}{s^2(s+\alpha)}=\frac{1}{2\alpha}\left[1+\frac{1}{\alpha}\ln(1+\alpha)\right].$$

Algorithm of solution

First Step. From inequality

$$\mathcal{X}(\varepsilon)=\frac{1}{2(\alpha-\varepsilon)}\left[\frac{1}{\varepsilon}\ln(1+\varepsilon)-\frac{1}{\alpha}\ln(1+\alpha)\right]\ge\frac{\gamma}{2}\qquad\text{the number}\qquad\varepsilon\in(0,\alpha)$$

is determined.

Second Step. We consider the following successive approximations:

$$\psi_{n+1}(x)=\frac{\int_0^x e^{-\alpha(x-t)}\psi_n(t)\,dt\cdot\left(e^{-mx}-e^{-qx}\right)}{\sqrt{x^2\left(\int_0^x e^{-\alpha(x-t)}\psi_n(t)\,dt\right)^2+\int_0^x e^{-\alpha(x-t)}\psi_n(t)\,dt+1}}+$$

$$+\int_0^\infty E_1(|x-t|)\sqrt{\frac{2}{\gamma}\left(\int_0^t e^{-\alpha(t-u)}\psi_n(u)\,du\right)\rho_\varepsilon(t)}\,dt,\ n=0,1,2,\ldots.$$

The initial approximation is $\psi_0(x)=e^{-\varepsilon x}$.

Third Step . The density of distribution function $f(x)$ is determined from formula (40).

Fourth Step. "Mean income" is determined from the following relation:

$$M=\int_0^R xf(x)\,dx,$$

where R is the maximum value of income.

The values of M at various α $\left(m=\frac{1}{2},\,q=\frac{1}{4}\right)$ are presented in the table for the case $\mu=1$.

Numerical calculations show (see Tab.) that the bigger is the degree of supercriticality (the ratio μ/α), the bigger is the income.

Table 1. "Mean income" for $\varepsilon=0,9$

α	0,2	0,4	0,6	0,8	1
M	517,978	160,175	58,825	25,461	12,81

Numerical calculations are performed with the use of MathCAD.

References

[1] I.D.Sargan, The distribution of wealth, *Econometrics*, **25**(1957), no. 4, 568–590.

[2] A.Kh.Khachatryan, Kh.A.Khachatryan, On the solvability of a nonlinear integro-differential equation arising in the income distribution problem, *Computational Mathematics and Mathematical Physics*, **50**(2010), no. 10, 1702–1711.

[3] B.Mandelbrot, The Pareto-Levy law and distribution of income, *International Econ. Rev.*, **1**(1960), no. 2, 79–106.

[4] V.M.Kakhktsyan, A.Kh.Khachatryan, Analytical-numerical solution of a nonlinear integrodifferential equation in econometrics, *Computational Mathematics and Mathematical Physics*, **53**(2013), no. 7, 933–936.

[5] A.Khachatryan, Kh.Khachatryan, On solvability of a nonlinear problem in theory of income distribution, *Eurasian Mathematical Journal*, **2**(2011), no. 2, 75–88.

[6] Kh.A.Khachatryan, On the solvability of an initial-boundary value problem for a nonlinear integro-differential equation with a noncompact operator of Hammerstein type, *Trudy Inst. Mat. i Mekh. UrO RAN*, **19**(2013), no. 3, 308–315 (in Russian).

[7] A.N.Kolmogorov, S.V.Fomin, Elements of the Theory of Functions and Functional Analysis, Paperback, 1999.

О разрешимости одного класса нелинейных интегро дифференциальных уравнений с некомпактным оператором Гаммерштейна, возникающим в теории распределения доходов

Агавард Х. Хачатрян
Хачатур А. Хачатрян
Тигран Г. Сардарян

В статье мы исследуем класс нелинейных интегро дифференциальных уравнений с некомпактным оператором Гаммерштейна, который имеет прямое применение в теории распределения доходов. Мы доказываем разрешимость класса уравнений в специальном весовом пространстве Соболева. Представлены также результаты численных вычислений.

Ключевые слова: оператор Гаммерштейна, весовое пространство Соболева, монотонность, итерации, условие Каратеодори.