УДК 519.218.2

On Long-time Behaviors of States of Galton-Watson Branching Processes Allowing Immigration

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Received 13.06.2015, received in revised form 01.08.2015, accepted 02.09.2015

We observe the discrete-time Branching Process allowing Immigration. Limit properties of transition functions and their convergence to invariant measures are investigated. In the critical situation a speed of this convergence is defined.

Keywords: branching process, immigration, transition functions, invariant measures, ratio limit property, rate of convergence.
DOI: 10.17516/1997-1397-2015-8-4-394-405

Introduction

Let the random function X_n denote the successive population size of the Galton-Watson Branching Process allowing Immigration (GWPI) at the moment $n \in \mathbb{N}_0$, here $\mathbb{N}_0 = \{0\} \cup \{\mathbb{N} = 1, 2, \ldots\}$. The state sequence $\{X_n\}$ is a homogeneous Markov chain with state space on $\mathbb{N}_0$ and can be expressed recursively as

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_{nk} + \eta_n, \quad for \quad n \in \mathbb{N},$$

where independent and identically distributed (i.i.d.) random variables ξ_{nk} denote the offspring number of k-th individual in the $(n-1)$-th generation, and i.i.d. variables η_n are not depend on ξ_{nk} interpreted as number of immigrants-individuals at the moment n. We assume $X_0 = 0$ and the process starts owing to immigrants. Each individual reproduces independently of each other and according to the offspring law $p_k := \mathbb{P}\{\xi_{11} = k\}$. With probability $h_j := \mathbb{P}\{\eta_1 = j\}$ arrive $j \in \mathbb{N}_0$ immigrants in population in each moment $n \in \mathbb{N}$. These individuals undergo further transformation by the reproduction law $\{p_j\}$. Throughout the paper we assume $p_0 > 0$ and $\sum_{j \in \mathbb{N}_0} h_j = 1$.

We denote $\mathcal{S} \subseteq \mathbb{N}_0$ to be the state space of the chain $\{X_n\}$. It is indicated by n-step transition functions

$$p_{ij}^{(n)} = \mathbb{P}_i\{X_n = j\} := \mathbb{P}\{X_{k+n} = j | X_k = i\},$$

for any $n, k \in \mathbb{N}_0$. Let

$$\mathcal{P}_n^{(i)}(s) := \mathbb{E}_i s^{X_n} = \sum_{j \in \mathcal{S}} p_{ij}^{(n)} s^j$$

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is probability generating function (PGF). Denoting

$$G(s) := \sum_{j\in\mathbb{N}_0} h_j s^j \qquad \text{and} \qquad F(s) := \sum_{j\in\mathbb{N}_0} p_j s^j,$$

one can see

$$\mathcal{P}^{(i)}_{n+1}(s) = G(s)\cdot\mathcal{P}^{(i)}_{n}\left(F(s)\right). \tag{0.1}$$

From (1.1) we have

$$\mathcal{P}^{(i)}_{n}(s) = \left[F_n(s)\right]^i \prod_{k=0}^{n-1} G\left(F_k(s)\right), \tag{0.2}$$

where $F_n(s)$ is n-fold iterate of PGF $F(s)$; see, e.g., [1, p. 263]. Now it is clear the probabilities $\left\{p^{(n)}_{ij}\right\}$ are completely defined by means of probabilities $\{p_j\}$ and $\{h_j\}$.

Classification of states of the chain $\{X_n\}$ is one of fundamental problems in theory of GWPI. Direct differentiation of (1.2) gives us

$$\mathbb{E}_i X_n = \sum_{j\in\mathcal{S}} j p^{(n)}_{ij} = \begin{cases} \left(\dfrac{\alpha}{A-1}+i\right)A^n - \dfrac{\alpha}{A-1}\ , & \text{when } A\neq 1,\\ \alpha n + i\ , & \text{when } A = 1,\end{cases}$$

where $A = F'(1)$ and $\alpha = G'(1)$. The received formula for $\mathbb{E}_i X_n$ shows that classification of states of GWPI depends on the value of parameter A is the mean number of direct descendants of single individual as a result of transformation for one-step generation. Process $\{X_n\}$ is classified as sub-critical, critical and supercritical if $A < 1$, $A = 1$ and $A > 1$ accordingly.

The above described evolution process of individuals was considered first by Heathcote [3] in 1965. Further long-term properties of states and a problem of existence and uniqueness of invariant measures of GWPI were investigated in papers of Seneta [8, 10, 11], Pakes [4–7] and by many other authors. Therein some moment conditions for PGF $F(s)$ and $G(s)$ was required to be satisfied. In aforementioned works of Seneta the ergodic properties of $\{X_n\}$ were investigated. He has proved that in cases $A \leqslant 1$ there is a unique invariant measure $\{\mu_k,\ k\in\mathcal{S}\}$ and besides $\mu_0 = 1$. Heathcote [2] and Pakes [7] have shown that in supercritical case $\mathcal{S}$ is transient. In the critical case if the first moment of immigration law $\alpha := G'(1)$ is finite, then $\mathcal{S}$ can be transient, null-recurrent or ergodic. In this case, if in addition to assume that $2B := F''(1) < \infty$, properties of $\mathcal{S}$ depend on value of parameter $\lambda = \alpha/B$: if $\lambda > 1$ or $\lambda < 1$, then $\mathcal{S}$ is transient or null-recurrent accordingly. In the case when $\lambda = 1$, Pakes [6] and Zubkov [15] studied necessary and sufficient conditions for a null-recurrence property. Limiting distribution law for critical process $\{X_n\}$ was found first by Seneta [9]. By them it has been proved under the condition of $0 < \lambda < \infty$ the normalized process X_n/n has limiting Gamma distribution with density function

$$\frac{1}{B\Gamma(\lambda)}\left(\frac{x}{B}\right)^{\lambda-1} e^{-x/B}, \quad \text{for } x > 0,$$

where $\Gamma(*)$ is Euler's Gamma function. This result without reference to Seneta has been established also by Pakes [6].

More recent researches on asymptotic properties of process contain in papers [12–14] in which the Bernoulli type GWPI was considered, i.e. both ξ_{nk} and η_n obey the Binomial distribution law. Clearly Bernoulli type GWPI is a special class in the general theory of Branching Processes. In this paper we consider processes in which both offspring law and immigration law are arbitrary.

In Section 2 invariant properties of GWPI will be investigated. The analogue of the Ratio Limit Property Theorem for transition functions $\left\{p_{ij}^{(n)}\right\}$ will be proved (Theorem 1 below).

The Section 3 is devoted to estimate of speed of convergence of $\left\{n^{\lambda}p_{ij}^{(n)}\right\}$ to invariant measures in the critical case.

1. Invariant property of transition functions

First we are interested in long-time behavior of ratio $p_{ij}^{(n)} \big/ p_{00}^{(n)}$ for any $i,j \in \mathcal{S}$. Having designation $\mathcal{P}_n(s) := \mathcal{P}_n^{(0)}(s)$, it follows from (1.2) that

$$\frac{\mathcal{P}_n^{(i)}(s)}{\mathcal{P}_n(s)} \longrightarrow q^i, \quad \text{as} \quad n \to \infty, \tag{1.1}$$

because of $F_n(s) \to q$ for $0 \leqslant s < 1$; see [17, p. 53]. Recall that q is an extinction probability of the simple branching process without immigration with PGF $F(s)$. It is the least nonnegative solution of $q = F(q)$, and that $q = 1$ if $A \leqslant 1$ and $q < 1$ if $A > 1$. Putting $s = 0$ in (2.1) implies $p_{i0}^{(n)} \big/ p_{00}^{(n)} \to q^i$ as $n \to \infty$. On purpose to receive the statement generally for all $j \in \mathcal{S}$, we write

$$\mathcal{P}_{n+1}(s) = \mathcal{P}_n(s) \cdot G\left(F_n(s)\right)$$

and from here and considering the properties of PGF one can calculate derivatives of j-th order:

$$\frac{\partial^j \mathcal{P}_{n+1}(s)}{\partial s^j} = \frac{\partial^j \mathcal{P}_n(s)}{\partial s^j} \cdot G\left(F_n(s)\right) + D_{j,n}(s),$$

for all $0 \leqslant s < 1$, where expression $D_{j,n}(s)$ is a power series with nonnegative coefficients. Since $p_{0j}^{(n)} = \partial^j \mathcal{P}_{n+1}(s) / \partial s^j \big|_{s=0}$, from last received results we obtain

$$\frac{p_{0j}^{(n+1)}}{p_{00}^{(n+1)}} \geqslant \frac{p_{0j}^{(n)}}{p_{00}^{(n)}}.$$

So the sequence of functions $\left\{p_{0j}^{(n)} \big/ p_{00}^{(n)}\right\}$ monotonously increases as $n \to \infty$. In our conditions $p_{00}^{(n)} > 0$ for any $n \in \mathbb{N}$. Therefore this sequence converges increasing to the finite non-negative limit which we will designate as υ_j:

$$\frac{p_{0j}^{(n)}}{p_{00}^{(n)}} \uparrow \upsilon_j < \infty, \quad \text{as} \quad n \to \infty. \tag{1.2}$$

Let's consider now more general ratio $p_{ij}^{(n)} \big/ p_{00}^{(n)}$. Denoting

$$\mathcal{U}_n^{(i)}(s) := \sum_{j \in \mathcal{S}} \frac{p_{ij}^{(n)}}{p_{00}^{(n)}} s^j, \quad \text{for} \quad 0 \leqslant s < 1,$$

we write the following equalities:

$$\mathcal{U}_n^{(i)}(s) = \sum_{j \in \mathcal{S}} \frac{p_{ij}^{(n)}}{p_{00}^{(n)}} s^j = \left[F_n(s)\right]^i \frac{\mathcal{P}_n(s)}{\mathcal{P}_n(0)} = \left[F_n(s)\right]^i \mathcal{U}_n(s), \tag{1.3}$$

where

$$\mathcal{U}_n(s) = \sum_{j\in\mathcal{S}} \frac{p_{0j}^{(n)}}{p_{00}^{(n)}} s^j.$$

Now wee prove the following Ratio Limit Property (RLP) Theorem.

Theorem 1. *The general GWPI satisfies the RLP for all $i, j \in \mathcal{S}$:*

$$\lim_{n\to\infty} \frac{p_{ij}^{(n)}}{p_{00}^{(n)}} = q^i \upsilon_j < \infty. \tag{1.4}$$

An appropriate PGF of $\upsilon_j = \lim_{n\to\infty} p_{0j}^{(n)} \Big/ p_{00}^{(n)}$ is

$$\mathcal{U}(s) = \sum_{j\in\mathcal{S}} \upsilon_j s^j$$

and it satisfies the functional equation

$$\sigma \cdot \mathcal{U}(s) = G(s) \cdot \mathcal{U}\left(F(s)\right), \tag{1.5}$$

in a region of its convergence, where $\sigma := G(q)$.

Proof. The statement (2.4) immediately follows from relations (2.2), (2.3) and that fact $F_n(s) \to q$ uniformly for $0 \leqslant s \leqslant r < 1$ as $n \to \infty$.

To prove the justice of the equation (2.5) we consider together the relations (1.1), (2.3) and the known equality $\mathcal{P}_{n+1}(s) = \mathcal{P}_n(s) \cdot G\left(F_n(s)\right)$ and receive the following equalities:

$$\begin{aligned} \mathcal{U}_{n+1}^{(i)}(s) &= \left[F_{n+1}(s)\right]^i \mathcal{U}_{n+1}(s) = \left[F_n\left(F(s)\right)\right]^i \frac{\mathcal{P}_{n+1}(s)}{\mathcal{P}_{n+1}(0)} = \\ &= \left[F_n\left(F(s)\right)\right]^i \frac{G(s) \cdot \mathcal{P}_n\left(F(s)\right)}{G\left(F_n(0)\right) \cdot \mathcal{P}_n(0)} = \frac{G(s)}{G\left(F_n(0)\right)} \cdot \mathcal{U}_n^{(i)}\left(F(s)\right). \end{aligned}$$

Taking limit as $n \to \infty$ from here we get to (2.5).

The Theorem is proved. □

PGF $\mathcal{U}(s)$ as a power series represents a continuous function in field of $0 \leqslant s < 1$. According to properties of PGF it converges for all $s \in [0;\, 1-\varepsilon]$ and for any arbitrary small constant $\varepsilon > 0$.

Repeatedly using the iteration of PGF $F(s)$ in the equation (2.5) leads us to the following relation:

$$\sigma^n \mathcal{U}(s) = \mathcal{P}_n(s) \mathcal{U}\left(F_n(s)\right). \tag{1.6}$$

The transition function analogue of (2.6) is

$$\sigma^n \cdot \upsilon_j = \sum_{i\in\mathcal{S}} \upsilon_i p_{ij}^{(n)}. \tag{1.7}$$

Equality (2.7) indicates that the set of non-negative numbers $\{\upsilon_j,\, j \in \mathcal{S}\}$ represents an invariant measure for the chain $\{X_n\}$.

Due to the condition $p_{00}^{(n)} > 0$ and the equality (2.7), all of $\upsilon_j < \infty$ and $\upsilon_j > 0$ for $j \in \mathcal{S}$. And $\upsilon_0 = 1$ as well. Then by definition of the process $\{X_n\}$ and owing to (2.7) we have the following chain of equalities:

$$\begin{aligned} \sigma^n &= \sigma^n \cdot \upsilon_0 = \sum_{i\in\mathcal{S}} \upsilon_i p_{i0}^{(n)} = \\ &= \sum_{i\in\mathcal{S}} \upsilon_i P_{i0}(n) p_{00}^{(n)} = p_{00}^{(n)} \sum_{i\in\mathcal{S}} \upsilon_i P_{10}^i(n), \end{aligned}$$

where $P_{i0}(n) = \mathbb{P}_i\{Z_n = 0\}$ is a hitting probability to zero state of the process $\{Z_n\}$ without immigration and generated by PGF $F(s)$. Since this probability is equal to $F_n(0)$, one can see $\sigma^n = \mathcal{P}_n(0)\mathcal{U}(F_n(0))$ and hence

$$\mathcal{U}(F_n(0)) = \frac{\sigma^n}{p_{00}^{(n)}}, \tag{1.8}$$

for any $n \in \mathbb{N}$.

Let's consider the case $A \neq 1$. Due to continuity of $\mathcal{U}(s)$, from equality (2.8) we receive

$$\frac{\sigma^n}{p_{00}^{(n)}} \longrightarrow \mathcal{U}(q) < \infty, \quad \text{as} \quad n \to \infty. \tag{1.9}$$

Here we considered that $F_n(0) \to q$. Now considering together the relations (2.1), (2.4) and (2.9), we can write the following theorem.

Theorem 2. *If $A \neq 1$, then*

$$\sigma^{-n} p_{ij}^{(n)} \longrightarrow \frac{q^i \upsilon_j}{\sum\limits_{k \in \mathcal{S}} q^k \upsilon_k}, \quad as \quad n \to \infty,$$

for all $i, j \in \mathcal{S}$, where $\sigma = G(q)$ and $\upsilon_j = \lim\limits_{n\to\infty} p_{0j}^{(n)} \Big/ p_{00}^{(n)}$.

Further we expand our discussion concerning the equation (2.5) investigating properties of its solution.

Theorem 3. *Let $A \neq 1$. Then there is a unique (up to a multiplicative constant) solution $\mathcal{U}(s)$ of the equation (2.5) for $s \in [0;\, q)$ such that*

$$\mathcal{L}(t) = \mathcal{U}(q - t) \tag{1.10}$$

is a slowly varying function as $t \downarrow 0$.

Proof. Let's propose that there is another solution $\widehat{\mathcal{U}}(s)$ of the equation (2.5). Then owing to equality (2.6) we write

$$\frac{\mathcal{U}(s)}{\widehat{\mathcal{U}}(s)} = \frac{\mathcal{U}(F_n(s))}{\widehat{\mathcal{U}}(F_n(s))}. \tag{1.11}$$

By definition the solution $\widehat{\mathcal{U}}(s)$ as well as $\mathcal{U}(s)$ monotonically increases. Since $F_n(0) \uparrow q$ then for given each $s \in [0;\, q)$ always there is $k \in \mathbb{N}$ such that $F_k(0) \leqslant s < F_{k+1}(0)$. Hence from equality (2.11) we will receive the following relations:

$$\frac{\mathcal{U}(s)}{\widehat{\mathcal{U}}(s)} \leqslant \frac{\mathcal{U}(F_{n+k+1}(0))}{\widehat{\mathcal{U}}(F_{n+k}(0))} = \frac{\mathcal{U}(F_{n+k+1}(0))}{\widehat{\mathcal{U}}(F_{n+k+1}(0))} \cdot \frac{\widehat{\mathcal{U}}(F_{n+k+1}(0))}{\widehat{\mathcal{U}}(F_{n+k}(0))}.$$

But again according to equality (2.11)

$$\frac{\mathcal{U}(F_n(0))}{\widehat{\mathcal{U}}(F_n(0))} = \frac{\mathcal{U}(0)}{\widehat{\mathcal{U}}(0)} = 1.$$

Then using once again (2.6) and the formula $\mathcal{P}_{n+1}(s) = \mathcal{P}_n(s) \cdot G(F_n(s))$, we have

$$\frac{\mathcal{U}(s)}{\widehat{\mathcal{U}}(s)} \leqslant \frac{\widehat{\mathcal{U}}(F_{n+k+1}(0))}{\widehat{\mathcal{U}}(F_{n+k}(0))} = \frac{\sigma}{G(F_{n+k}(0))}.$$

Taking limit as

$$\frac{\mathcal{U}(s)}{\widehat{\mathcal{U}}(s)} \leqslant 1$$

because PGF $G(s)$ continuously. By the similar way it is possible to establish the converse inequality $\mathcal{U}(s)\Big/\widehat{\mathcal{U}}(s) \geqslant 1$. The received conclusions say that the equation (2.5) has a unique solution for all $s \in [0;\, q)$.

Now following a method of Seneta [8] we put

$$g(s) = G(q - s), \quad f(s) = q - F(q - s).$$

In the allowed designations the equation (2.5) becomes

$$\sigma \cdot \mathcal{L}(s) = g(s) \cdot \mathcal{L}\left(f(s)\right), \quad for \quad s \in [0;\, q). \tag{1.12}$$

We will be convinced the function $f(s)/s$ monotonically decreases on set of $0 \leqslant s < q$ taking here the maximum $\beta = \lim_{s\downarrow 0}\left[f(s)/s\right]$ and the minimum $f(q)/q = 1 - p_0/q$ accordingly, where as before $\beta = F'(q)$. The function $\mathcal{L}(s)$ in form of (2.10) is also monotonically decreasing on this set and $\lim_{s\downarrow 0} g(s) = \sigma$. Then for any $\lambda \in [\beta;\, 1]$ one gets

$$\frac{\mathcal{L}\left(f(s)\right)}{\mathcal{L}(s)} = \frac{\mathcal{L}\left(\dfrac{f(s)}{s}s\right)}{\mathcal{L}(s)} \geqslant \frac{\mathcal{L}(\beta s)}{\mathcal{L}(s)} \geqslant \frac{\mathcal{L}(\lambda s)}{\mathcal{L}(s)} \geqslant 1.$$

On the other hand according to (2.12),

$$\frac{\mathcal{L}\left(f(s)\right)}{\mathcal{L}(s)} = \frac{\sigma}{g(s)} \longrightarrow 1, \quad as \quad s \downarrow 0.$$

Hence,

$$\lim_{s\downarrow 0} \frac{\mathcal{L}(\lambda s)}{\mathcal{L}(s)} = 1,$$

for any $\lambda \in [\beta;\, 1]$. It is easy to be convinced that the last relation is valid for any $\lambda \in \mathbb{R}_+$, where $\mathbb{R}_+$ is set of positive real numbers. So $\mathcal{L}(s) = \mathcal{U}(q - s)$ is a slowly varying function as $s \downarrow 0$.

The Theorem is proved. □

In the critical case it has been proved by Pakes [5] that the sequence $\left\{n^{\lambda}\mathcal{P}_n(s)\right\}$ converges to the limiting PGF $\pi(s)$ uniformly for $0 \leqslant s \leqslant r < 1$ which is a solution of the equation (2.5):

$$\pi(s) = G(s)\pi\left(F(s)\right),$$

where as before $\lambda = \alpha/B$. It was supposed therein that the moments

$$\sum_{j\in\mathcal{S}} p_j j^2 \ln j \qquad \text{and} \qquad \sum_{j\in\mathcal{S}} h_j j \ln j$$

are finite. An advantage of assertion of the Theorem 1 from aforementioned result of Pakes consists that in our case the invariant measure $\{\upsilon_j\}$ for GWPI and corresponding for it the equation (2.5) is established without any moment assumptions concerning distributions $\{p_j\}$ and $\{h_j\}$. In the final Section of the paper we investigate a speed of convergence

$$n^{\lambda}\mathcal{P}_n(s) \longrightarrow \pi(s), \quad \text{as} \quad n \to \infty,$$

strengthening aforementioned result of Pakes.

2. A speed rate of convergence to invariant measures in critical situation

Consider the critical GWPI with transition functions $p_{ij}^{(n)} = \mathbb{P}_i \{X_n = j\}$. Recall the appropriate PGF

$$\mathcal{P}_n(s) = \sum_{j \in \mathcal{S}} p_{0j}^{(n)} s^j = \prod_{k=0}^{n-1} G(F_k(s)), \tag{2.1}$$

where $G(s)$ and $F(s)$ are PGF's of immigration stream law and the process offspring law accordingly. Provided that moments $G''(1)$ and $F^{IV}(1)$ are finite, Pakes [5] investigated a rate of convergence $\{n^\lambda \mathcal{P}_n(s)\}$ to the limiting PGF $\pi(s) = \sum_{j \in \mathcal{S}} \pi_j s^j$, which is being the solution of functional equation $\pi(s) = G(s) \cdot \pi(F(s))$. So the nonnegative numbers $\{\pi_j\}$ satisfy the relation

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij}^{(n)}.$$

In this section we improve aforementioned result of Pakes, holding to condition of the third order factorial moment of offspring PGF is finite.

Theorem 4. *Let $A = 1$, $2B := F''(1)$, $\alpha := G'(1)$ and $\lambda = \alpha/B$. If $C := F'''(1) < \infty$, then the sequence $\{n^\lambda \mathcal{P}_n(s)\}$ converges to $\pi(s)$ uniformly for $0 \leqslant s \leqslant r < 1$, and besides*

$$n^\lambda \mathcal{P}_n(s) = \pi(s) \cdot \left(1 + \Delta \cdot \frac{\ln b_n(s)}{b_n(s)} (1 + o(1))\right), \quad as \;\; n \to \infty, \tag{2.2}$$

where $\Delta := \alpha \left(\dfrac{C}{6B^2} - 1\right)$ and

$$b_n(s) = Bn + \frac{1}{1-s}.$$

Proof. It follows from (3.1) that

$$\begin{aligned} n^\lambda \mathcal{P}_n(s) &= n^\lambda \prod_{k=0}^{n-1} G(F_k(s)) = \\ &= G(s) \prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^\lambda G(F_k(s)) = G(s) \prod_{k=1}^{n-1} A_k(s), \end{aligned} \tag{2.3}$$

where $A_k(s) = \left(1 + \dfrac{1}{k}\right)^\lambda G(F_k(s))$. It is known that the infinite product $\prod_{k \in \mathbb{N}} A_k(s)$ and the series $\sum_{k \in \mathbb{N}} (A_k(s) - 1)$ converge or diverge simultaneously. Therefore we investigate the last series. Using elementary expansion $(1 + 1/k)^\lambda = 1 + \lambda/k + \varepsilon_k$, we have the following representation:

$$A_k(s) - 1 = \frac{\lambda}{k} - (1 - G(F_k(s))) - \frac{\lambda}{k} (1 - G(F_k(s))) + \varepsilon_k G(F_k(s)), \tag{2.4}$$

where $\varepsilon_k = O\left(1/k^2\right)$. We write

$$1 - G(s) = \alpha \cdot (1 - s) - \delta(s)(1 - s), \tag{2.5}$$

where $0 \leqslant \delta(s) = (1-s)\, G''(\theta)/2$ and $s < \theta < 1$; obviously that $\delta(s) = o(1)$ as $s \uparrow 1$. In turn we know that (see [1, p.74]):

$$|1 - F_n(s)| \leqslant 2\,(1 - F_n(0))\,, \tag{2.6}$$

and according to the Basic Lemma of the theory of critical processes $1 - F_n(0) \sim 1/Bn$; see e.g. [1, p.19]. Hence from relations (3.4)–(3.6) we will easily be convinced that

$$A_k(s) - 1 = O\left(\frac{1}{k^2}\right), \quad \text{as} \quad k \to \infty,$$

uniformly for $0 \leqslant s \leqslant r < 1$. Last statement testifies to a uniform convergence of the series $\sum_{k\in\mathbb{N}} (A_k(s) - 1)$ and hence the infinite product $G(s) \prod_{k\in\mathbb{N}} A_k(s)$. We denote this as

$$\pi(s) := \lim_{n\to\infty} n^{\lambda} \mathcal{P}_n(s) = G(s) \prod_{k\in\mathbb{N}} A_k(s). \tag{2.7}$$

Now to the proof of relation (3.2), we will estimate the error term of difference $n^{\lambda}\mathcal{P}_n(s) - \pi(s)$. So using representation (3.3) and equality (3.7) we obtain

$$\begin{aligned} n^{\lambda}\mathcal{P}_n(s) - \pi(s) &= G(s) \cdot \left[\prod_{k=1}^{n-1} A_k(s) - \prod_{k=1}^{\infty} A_k(s)\right] = \\ &= G(s) \prod_{k=1}^{n-1} A_k(s) \cdot \left[1 - \prod_{k=n}^{\infty} A_k(s)\right]. \end{aligned} \tag{2.8}$$

According to the positiveness property of PGF we see $A_k(s) > 0$ for all $0 \leqslant s < 1$. Then using the elementary inequality $\ln(1-x) \geqslant -x - x^2/(1-x)$ we write down the following equalities:

$$\begin{aligned} \ln \prod_{k\geqslant n} A_k(s) &= \sum_{k\geqslant n} \ln\{1 - (1 - A_k(s))\} = \\ &= -\sum_{k\geqslant n} (1 - A_k(s)) + \rho_n^{(1)}(s) =: \Sigma_n(s) + \rho_n^{(1)}(s), \end{aligned} \tag{2.9}$$

where

$$\Sigma_n(s) = -\sum_{k\geqslant n} (1 - A_k(s)),$$

and

$$\begin{aligned} 0 \geqslant \rho_n^{(1)}(s) \quad &\geqslant \quad -\sum_{k\geqslant n} \frac{[1 - A_k(s)]^2}{A_k(s)} > \\ &> \quad -(1 - G(F_n(s))) \sum_{k\geqslant n} \frac{1 - A_k(s)}{A_k(s)} > \\ &> \quad \frac{1 - G(F_n(s))}{G(F_n(s))} \cdot \Sigma_n(s). \end{aligned}$$

The monotone property of PGFs used in the last step.

Replacing s by $F_k(s)$ it follows from (3.5) that

$$1 - G(F_n(s)) = \alpha\,(1 - F_n(s)) + \delta\,(F_n(s))\,(1 - F_n(s))\,. \tag{2.10}$$

Owing to (3.6) and (3.10) $1-G\left(F_n(s)\right) \sim \lambda/n$ and hence the augend in (3.9) $\rho_n^{(1)}(s) \to 0$ always supposing the first term $\Sigma_n(s)$ has a finite limit as $n \to \infty$. In turn in our conditions and owing to (3.6) $\delta\left(F_n(s)\right) = O\left(1/n\right)$. Therefore combining (3.4), (3.6) and (3.10) we will receive the following equality for first term in (3.9):

$$\Sigma_n(s) = -\sum_{k \geqslant n} \left(\alpha\left(1-F_k(s)\right) - \frac{\lambda}{k}\right) + \sum_{k \geqslant n} O\left(\frac{1}{k^2}\right). \tag{2.11}$$

Further we use the following asymptotic expansion for the function $1-F_n(s)$ which holds in the conditions of our theorem:

$$1-F_n(s) = \frac{1}{b_n(s)} + \widetilde{\Delta} \cdot \frac{\ln b_n(s) + K(s)}{\left(b_n(s)\right)^2}\left(1+o(1)\right), \tag{2.12}$$

as $n \to \infty$, where $\widetilde{\Delta} = \dfrac{C}{6B} - B$ and $K(s)$ is some bounded function depending on form of $F(s)$ and $b_n(s)$ is same as in the theorem statement. The formula (3.12) was established in the paper [16] and we have reduced it in a bit modified form. So $b_n(s) = O(n)$ as $n \to \infty$, considering the expansion (3.12), we rewrite the equality (3.11) in form of

$$\Sigma_n(s) = -\alpha\widetilde{\Delta} \cdot \sum_{k \geqslant n} \chi(k) + \rho_n^{(2)}(s), \tag{2.13}$$

where $\rho_n^{(2)}(s) = \sum_{k \geqslant n} O\left(1/k^2\right)$ and

$$\chi(k) = \frac{\ln b_k(s)}{\left(b_k(s)\right)^2}.$$

One can see that the function $\chi(k)$ is positive and monotonically decreases with respect $k \in \mathbb{N}$ and for all $0 \leqslant s < 1$.

We consider now an alternative function $\chi(t)$ for $t \in \mathbb{R}_+$. Obviously this function is positive, monotonically decreases and also is continuous. Moreover

$$\int \chi(t)dt = \mathrm{X}(t) + \mathrm{const},$$

where

$$\mathrm{X}(t) = -\frac{1}{B}\left(\frac{\ln b_t(s)}{b_t(s)} + \frac{1}{b_t(s)}\right),$$

for $0 \leqslant s < 1$ and $\mathrm{X}(t) \to 0$ as $t \to \infty$. Therefore due to the Mc'Loren-Cauchy test (see [18, pp. 283–284]) the following inequalities hold:

$$\frac{1}{B}\frac{\ln b_n(s)}{b_n(s)} + \frac{1}{b_n(s)} \leqslant \sum_{k \geqslant n} \chi(k) \leqslant \frac{1}{B}\frac{\ln b_{n-1}(s)}{b_{n-1}(s)} + \frac{1}{b_{n-1}(s)}.$$

By means of the last inequalities and considering that $b_n(s) = O(n)$ as $n \to \infty$ we write estimation

$$\left|\sum_{k \geqslant n} \chi(k) - \frac{1}{B}\frac{\ln b_n(s)}{b_n(s)}\right| = O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty, \tag{2.14}$$

for $0 \leqslant s < 1$.

Almost obviously that $\rho_n^{(2)}(s) = \sum_{k \geqslant n} O\left(1/k^2\right) = O\left(1/n\right)$. Considering it and exploiting (3.14) in (3.13) we will obtain

$$\Sigma_n(s) = -\Delta \cdot \frac{\ln b_n(s)}{b_n(s)} + O\left(\frac{1}{n}\right), \quad \text{as} \ \ n \to \infty, \tag{2.15}$$

where $\Delta = \alpha \left(\dfrac{C}{6B^2} - 1\right)$.

Return now to the equality (3.9). Due to (3.15) and the fact $1 - G\left(F_n(s)\right) \sim \lambda/n$ arises $\rho_n^{(1)}(s) = O\left(\ln n/n^2\right)$. Then from (3.9) and (3.15) we conclude

$$\prod_{k \geqslant n} A_k(s) = \exp\left\{-\Delta \cdot \frac{\ln b_n(s)}{b_n(s)}\left(1 + o\left(1\right)\right)\right\}, \quad \text{as} \ \ n \to \infty.$$

Finally using last expansion in (3.8) with combination of the formula $1 - e^{-x} \sim x, \ x \to 0$, we complete the theorem proof. □

From Theorem 4 we receive the following

Corollary 1. *In conditions of Theorem 4 the following assertion is valid*

$$n^{\lambda} p_{00}^{(n)} = \pi_0 \left(1 + \frac{\Delta}{B} \cdot \frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)\right), \quad \text{as} \ \ n \to \infty,$$

where Δ *is defined in Theorem 4.*

The following theorem generalizes the previous one.

Theorem 5. *Let conditions of the Theorem 4 are satisfied. Then the sequence* $\left\{n^{\lambda}\mathcal{P}_n^{(i)}(s)\right\}$ *converges to the limiting function* $\pi(s)$ *uniformly on the set of* $0 \leqslant s \leqslant r < 1$, *and*

$$n^{\lambda}\mathcal{P}_n^{(i)}(s) = \pi(s)\left(\delta_n^{(i)}(s) + \Delta_n^{(i)}(s) \cdot \frac{\ln b_n(s)}{b_n(s)}\left(1 + o\left(1\right)\right)\right), \tag{2.16}$$

as $n \to \infty$, *where* $\Delta_n^{(i)}(s) = \Delta \cdot \delta_n^{(i)}(s)$ *and*

$$\delta_n^{(i)}(s) = 1 - \frac{i}{b_n(s)},$$

and expressions Δ *and* $b_n(s)$ *are defined in Theorem 4.*

Proof. Since $F_n(s) \leqslant 1$ and $F_n(s) \uparrow 1$ as $n \to \infty$, it follows from (1.2) and Theorem 4 that $\left\{n^{\lambda}\mathcal{P}_n^{(i)}(s)\right\}$ converges uniformly to $\pi(s)$ for $0 \leqslant s \leqslant r < 1$. Write

$$n^{\lambda}\mathcal{P}_n^{(i)}(s) = \left(F_n(s)\right)^i n^{\lambda}\mathcal{P}_n(s). \tag{2.17}$$

It is obvious for fixed i and at large values of number n

$$\left(F_n(s)\right)^i = 1 - i\left(1 - F_n(s)\right)\left(1 + o(1)\right).$$

From here and using (3.12) follows

$$\left(F_n(s)\right)^i = 1 - \frac{i}{b_n(s)}\left(1 + o(1)\right), \quad \text{as} \ \ n \to \infty. \tag{2.18}$$

Now the theorem statement follows from equalities (3.17) and (3.18), with application of the statements (3.2) and (3.12). □

Corollary 2. *In conditions of Theorem 5 the following assertion is valid:*

$$n^{\lambda} p_{ij}^{(n)} = \pi_j \left(\delta_n^{(i)} + \Delta\delta_n^{(i)} \cdot \frac{\ln Bn}{Bn} (1 + o(1)) \right), \quad as \quad n \to \infty,$$

for all $i, j \in \mathcal{S}$*, where* $\delta_n^{(i)} = 1 - i/Bn$.

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Об асимптотическом поведении состояний ветвящихся процессов Гальтона-Ватсона с иммиграцией

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В работе рассматривается ветвящийся процесс с иммиграцией дискретного времени. Исследуются предельные свойства переходных вероятностей и их сходимость к инвариантным мерам. В критическом случае определяется скорость этой сходимости.

Ключевые слова: ветвящийся процесс, иммиграция, переходные вероятности, инвариантные меры, скорость сходимости к инвариантным мерам.