Journal of Siberian Federal University. Mathematics & Physics 2014, 7(4), 455–465

УДК 517.55

# On Some Systems of Non-algebraic Equations in $\mathbb{C}^n$

**Olga V. Khodos***
Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny, 79, Krasnoyarsk, 660041
Russia

Received 06.05.2014, received in revised form 09.06.2014, accepted 11.08.2014

*A method of finding residue integrals for systems of non-algebraic equations containing entire functions is presented in the paper. Such integrals are connected with the power sums of roots of certain system of equations. The proposed approach can be used for developing methods for the elimination of unknowns from systems of non-algebraic equations. It is shown that obtained results can be used for investigation some model of chemical kinetics.*

*Keywords: non-algebraic systems of equations, residue integral, power sums.*

## Introduction

A method for the elimination $n$ unknowns from a system of $n$ non-linear algebraic equations (in the characteristic zero setting) based on multidimensional residue theory was proposed by L.Aizenberg [1]. Further developments of the method can be found in [2–4].

In general, the set of roots of a system of $n$ non-algebraic equations in $n$ variables is infinite. Moreover, multidimensional Newton series (with exponents in $\mathbb{N}^n$) of the roots of such systems is usually divergent. In the paper, we connect residue integrals with specific systems of $n$ non-linear equations and compute such residue integrals. Then we obtain from this computation (provided that such series do converge) the values of the sums of multidimensional Newton series (with exponents in $(-\mathbb{N}^*)^n$) formed with the roots of such non-linear systems which do not belong to the union of coordinate planes.

A class of systems of equations containing entire or meromorphic functions was considered in [5].

The purpose of this paper is to generalize results given in [5] to a wider class of systems of non-algebraic equations; to obtain formulas for calculation of residue integrals and to reveal the connection between residue integrals and multidimensional power sums of roots.

## 1. Preliminaries

A.Kytmanov and Z.Potapova [5] considered the following system of functions:

$$f_1(z),\ f_2(z), \dots, f_n(z),$$

where $z = (z_1, z_2, \dots, z_n)$. Each function $f_j(z)$ is analytic in the neighborhood of $0 \in \mathbb{C}^n$ and has the form

$$f_j(z) = z^{\beta^j} + Q_j(z), \quad j = 1, 2, \dots, n,$$

*OKhodos@sfu-kras.ru

where $\beta^j = (\beta^j_1, \beta^j_2, \dots, \beta^j_n)$ is a vector of integer nonnegative indices, $z^{\beta^j} = z_1^{\beta^j_1} \cdot z_2^{\beta^j_2} \cdots z_n^{\beta^j_n}$, and $\|\beta^j\| = \beta^j_1 + \beta^j_2 + \ldots + \beta^j_n = k_j$, $j = 1, 2, \dots, n$. Functions $Q_j$ are expanded in a neighborhood of zero into an absolutely and uniformly converging Taylor series of the form

$$Q_j(z) = \sum_{\|\alpha\| > k_j} a^j_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \geqslant 0$, $\alpha_j \in \mathbb{Z}$, and $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

The formulas for calculation of residue integrals

$$J_\beta = \frac{1}{(2\pi i)^n} \int_{\gamma(r)} \frac{1}{z^{\beta+U}} \cdot \frac{df}{f}$$

in terms of coefficients of $Q_j(z)$ were obtained.

Our goal is to obtain similar results in a more general case.

## 2. Calculation of residue integrals

We consider a system of functions $f_1(z),\ f_2(z), \dots, f_n(z)$. They are analytic in a neighborhood of the point $0 \in \mathbb{C}^n$, $z = (z_1, z_2, \dots, z_n)$ and has the form

$$f_j(z) = (z^{\beta^j} + Q_j(z)) e^{P_j(z)}, \quad j = 1, 2, \dots, n, \tag{1}$$

where $\beta^j = (\beta^j_1, \beta^j_2, \dots, \beta^j_n)$ is a vector of integer nonnegative indices $z^{\beta^j} = z_1^{\beta^j_1} \cdot z_2^{\beta^j_2} \cdots z_n^{\beta^j_n}$ and $\|\beta^j\| = \beta^j_1 + \beta^j_2 + \ldots + \beta^j_n = k_j$, $j = 1, 2, \dots, n$. Functions $Q_j$, $P_j$ are expanded in a neighborhood of zero into an absolutely and uniformly converging Taylor series of the form

$$Q_j(z) = \sum_{\|\alpha\| > k_j} a^j_\alpha z^\alpha, \tag{2}$$

$$P_j(z) = \sum_{\|\gamma\| \geqslant 0} b^j_\gamma z^\gamma, \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \geqslant 0$, $\alpha_j \in \mathbb{Z}$, and $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$; $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, $\gamma_j \geqslant 0$, $\gamma_j \in \mathbb{Z}$, and $z^\gamma = z_1^{\gamma_1} \cdot z_2^{\gamma_2} \cdots z_n^{\gamma_n}$.

Firstly this system was considered in [6, 7].

So the degree of all monomials in $Q_j$ greater then $k_j$, $j = 1, \dots, n$.

Consider the integration cycles $\gamma(r) = \gamma(r_1, r_2, \dots, r_n)$, that are skeletons of the polydisks:

$$\gamma(r) = \{z \in \mathbb{C}^n : |z_s| = r_s, s = 1, 2, \dots, n\}, \quad r_1 > 0, \dots, r_n > 0.$$

For sufficiently small $r_j$, cycles $\gamma(r)$ lie in the domain where functions $f_j$ are analytic. Therefore, the series

$$\sum_{\|\alpha\| > k_j} |a^j_\alpha| r_1^{\alpha_1} \cdots r_n^{\alpha_n}$$

$$\sum_{\|\gamma\| \geqslant 0} |b^j_\gamma| r_1^{\gamma_1} \cdots r_n^{\gamma_n}$$

converge for $j = 1, 2, \dots, n$. Then, on the cycle $\gamma(tr) = \gamma(tr_1, tr_2, \dots, tr_n)$, $t > 0$, we have

$$|z|^{\beta^j} = t^{k_j} \cdot r_1^{\beta^j_1} \cdot r_2^{\beta^j_2} \cdots r_n^{\beta^j_n} = t^{k_j} \cdot r^{\beta^j}$$

and

$$|Q_j(z)| = \left| \sum_{\|\alpha\|>k_j} a^j_\alpha z^\alpha \right| \leqslant \sum_{\|\alpha\|>k_j} t^{\|\alpha\|} |a^j_\alpha| r^\alpha \leqslant t^{k_j+1} \sum_{\|\alpha\|>k_j} |a^j_\alpha| r^\alpha,$$

$$0 \leqslant t \leqslant 1, \quad j = 1, \dots, n.$$

Therefore, for sufficiently small positive $t$, the following inequalities hold on the cycle $\gamma(tr)$:

$$|z|^{\beta^j} > |Q_j(z)|, \quad j = 1, 2, \dots, n. \tag{4}$$

Thus,

$$f_j(z) \neq 0 \quad \text{on} \quad \gamma(tr), \quad j = 1, 2, \dots, n.$$

In what follows we assume that $t = 1$.

Consider the system of equations

$$\begin{cases} f_1(z) = 0, \\ f_2(z) = 0, \\ \dots\dots\dots\dots\dots \\ f_n(z) = 0. \end{cases} \tag{5}$$

In general, system (5) can have non-discrete set of roots.

It follows from (4) that for sufficiently small $r_j$ the following integrals exist:

$$\int\limits_{\gamma(r)} \frac{1}{z^{\beta+U}} \cdot \frac{df}{f} = \int\limits_{\gamma(r_1, r_2, \dots, r_n)} \frac{1}{z_1^{\beta_1+1} \cdot z_2^{\beta_2+1} \cdots z_n^{\beta_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_n}{f_n},$$

where $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0, \dots, \beta_n \geqslant 0$, $\beta_j \in \mathbb{Z}$, $U = (1, 1, \dots, 1)$. We call such integrals the *residue integrals*. These integrals are not the standard Grothendieck residues, since the cicle $\gamma(r)$ does not connect with fuctions $f_1, \dots, f_n$. The Logarithmic Residue Theorem is not applicable to such integrals as well.

These integrals do not depend on $(r_1, \dots, r_n)$ under condition (4) on $\gamma(r)$.

Let us introduce the following notations

$$J_\beta = \frac{1}{(2\pi i)^n} \int_{\gamma(r)} \frac{1}{z^{\beta+U}} \cdot \frac{df}{f}.$$

and $\widetilde{f}_j(z) = z^{\beta^j} + Q_j(z)$, $j = 1, \dots, n$.

Let us assume that $I^s$ is a vector of indices. The vector has $n$ components and consists of $s$ ones and $n - s$ zeros ($s = 0, \dots, n$). More exactly, each $I^s = I[i_1, \dots, i_s] = (0, \dots, 0, \overset{i_1}{1}, 0, \dots, 0, \overset{i_s}{1}, 0, \dots, 0) \in (\{0, 1\})^n$ where $i_1, \dots, i_s$ are the places of "one" in $I^s$, $1 \leqslant i_1 < \dots < i_s \leqslant n$. In what follows $\Delta_{I^s}$ stands for the Jacobian matrix of the system of functions such that to each "one" on the $j$-th place in $I^s$ there corresponds $j$-th row of the derivatives $\left(\partial \widetilde{f}_j / \partial z_i\right)$, $1 \leqslant i \leqslant n$ in $\Delta_{I^s}$ and to each "zero" on the $k$-th place in $I$ there corresponds $k$-th row of the derivatives $(\partial P_k / \partial z_i)$, $1 \leqslant i \leqslant n$ in $\Delta_{I^s}$.

**Theorem 1** ( [6,7]). *Under the assumptions made for the functions $f_j$ defined by (1), (2), (3) the following relations are valid:*

$$J_\beta = \sum_{s=0}^{n} \sum_{I^s} \sum_{\|\alpha^s\| \leqslant \|\beta\| + \min(s, k_{i_1} + \dots + k_{i_s})} \frac{(-1)^{\|\alpha^s\|}}{(\beta + (\alpha_1^s + 1)\beta^{i_1^s} + \dots + (\alpha_s^s + 1)\beta^{i_s^s})!} \times$$

$$\times \frac{\partial^{l_s}(\Delta_{I^s}\cdot Q^{\alpha^s}(I^s))}{\partial z^{\beta+(\alpha_1^s+1)\beta^{i_1^s}+\ldots+(\alpha_s^s+1)\beta^{i_s^s}}}\Bigg|_{z=0}$$

*or*

$$J_\beta=\sum_{s=0}^{n}\sum_{I^s}\sum_{\|\alpha^s\|\leqslant\|\beta\|+\min(n,k_{i_1}+\ldots+k_{i_s})}(-1)^{\|\alpha^s\|}\mathfrak{M}\left[\frac{\Delta_{I^s}\cdot Q^{\alpha^s}(I^s)}{z^{\beta+(\alpha_1^s+1)\beta^{i_1^s}+\ldots+(\alpha_s^s+1)\beta^{i_s^s}}}\right], \qquad (6)$$

*where $\alpha^s$ is a vector of indices with s components; $i_k^s$ is the index of the k-th 1 in $I^s$; $l_s=\|\beta+(\alpha_1^s+1)\beta^{i_1^s}+\ldots+(\alpha_s^s+1)\beta^{i_s^s}\|$; $\beta!=\beta_1!\cdot\beta_2!\cdots\beta_n!$; $Q^{\alpha^s}(I^s)=Q_{i_1^s}^{\alpha_1^s}\cdot Q_{i_2^s}^{\alpha_2^s}\cdots Q_{i_s^s}^{\alpha_s^s}$; $\dfrac{\partial^{\|\gamma\|}\varphi}{\partial z^\gamma}=\dfrac{\partial^{\gamma_1+\cdots\gamma_n}\varphi}{\partial z_1^{\gamma_1}\partial z_2^{\gamma_2}\cdots\partial z_n^{\gamma_n}}$; and $\mathfrak{M}$ is a linear functional that assigns constant term to a Laurent polynomial.*

**Remark 1**. *According to the proof the relation given in the statement of Theorem 1 contains only a finite number of coefficients of the functions $Q_j(z)$ and $P_j(z)$.*

**Corollary 1** ( [7]). *If all $\beta^j=(0,0,\ldots,0)$, $j=1,\ldots,n$, then the integral $J_\beta$ is*

$$J_\beta=\sum_{s=0}^{n}\sum_{I^s}\sum_{\|\alpha^s\|\leqslant\|\beta\|}(-1)^{\|\alpha\|}\mathfrak{M}\left[\frac{\Delta_{I^s}\,Q(I^s)^{\alpha^s}}{z^\beta}\right]=$$

$$=\sum_{s=0}^{n}\sum_{I^s}\sum_{\|\alpha^s\|\leqslant\|\beta\|}\frac{(-1)^{\|\alpha^s\|}}{\beta!}\frac{\partial^{\|\beta\|}}{\partial z^\beta}\left(\Delta_I\,Q(I)^{\alpha^s}\right)\Bigg|_{z=0}.$$

In the case of $\beta^j=(0,0,\ldots,0)$, it is also possible to obtain relation for $J_\beta$ with the use of the Cauchy integral formula for several complex variables, since $f_j(0)\neq 0$ for all $j=1,\ldots,n$.

## 3. Power sums

Our next goal is to connect considered above integrals with power sums of roots of system (5). We must reduce the class of functions $f_j$. At first we take $Q_j$ $(j=1,2,\ldots,n)$ as polynomials of the form

$$Q_j(z)=\sum_{\alpha\in M_j}a_\alpha^j z^\alpha, \qquad (7)$$

where $M_j$ is finite set of multi-indexes such that for $\alpha\in M_j$ coordinates $\alpha_k\leqslant\beta_k^j$, $k=1,2,\ldots,n$, $k\neq j$, but $\|\alpha\|>k_j$ for all $\alpha\in M_j$ as before. Functions $P_j$ $(j=1,2,\ldots,n)$ are polynomials of the form

$$P_j(z)=\sum_{0\leqslant\|\gamma\|\leqslant p_j}b_\gamma^j z^\gamma. \qquad (8)$$

Let us introduce the substitution $z_j=\dfrac{1}{w_j}$, $j=1,2,\ldots,n$. Therefore, we obtain

$$f_j\left(\frac{1}{w_1},\frac{1}{w_2},\ldots,\frac{1}{w_n}\right)=\left[\frac{1}{w^{\beta^j}}+Q_j\left(\frac{1}{w_1},\frac{1}{w_2},\ldots,\frac{1}{w_n}\right)\right]e^{P_j\left(\frac{1}{w_1},\frac{1}{w_2},\ldots,\frac{1}{w_n}\right)}=$$

$$=\frac{1}{w^{\beta^j+s_je^j}}\left(w_j^{s_j}+\widetilde{Q}_j(w_1,w_2,\ldots,w_n)\right)e^{P_j\left(\frac{1}{w_1},\frac{1}{w_2},\ldots,\frac{1}{w_n}\right)},$$

where $s_j$ is the degree of $w_j$, $e^1=(1,0,\ldots,0)$, $e^2=(0,1,\ldots,0)$, …, $e^n=(0,0,\ldots,1)$, and degree of polynomials

$$\widetilde{Q}_j(w_1,w_2,\ldots,w_n)=\widetilde{Q}_j(w)=w^{\beta^j+s_je^j}\cdot Q_j\left(\frac{1}{w_1},\frac{1}{w_2},\ldots,\frac{1}{w_n}\right)$$

is less than $s_j$.

According to the Bezout theorem the system of nonlinear algebraic equations

$$\widetilde{f}_j(w) = w_j^{s_j} + \widetilde{Q}_j(w) = 0, \quad j = 1, 2, \ldots, n, \tag{9}$$

has a finite number of roots that equals to $s_1 \cdot s_2 \cdots s_n$ and it has no roots on the infinite hyperplane $\mathbb{CP}^n \setminus \mathbb{C}^n$.

Let us denote roots of system (5) not lying on coordinate planes as $w_{(k)} = (w_{1(k)}, w_{2(k)}, \ldots, w_{n(k)})$, $k = 1, 2, \ldots, M$, $M \leqslant s_1 \cdot s_2 \cdots s_n$. Then points $z_{(k)} = \left(\frac{1}{w_{1(k)}}, \frac{1}{w_{2(k)}}, \ldots, \frac{1}{w_{n(k)}}\right)$ are the roots of system (5), not lying on coordinate planes. So we have the following assertion

**Lemma 1.** *System (5) with polynomials $Q_j$ of the form (7) and $P_j$ of the form (8) has a finite number of roots $z_{(1)}, z_{(2)}, \ldots, z_{(M)}$ not lying on coordinate planes $\{z_s = 0\}$, $s = 1, 2, \ldots, n$.*

Let us introduce notation

$$\sigma_{\beta+I} = \sigma_{(\beta_1+1, \beta_2+1, \ldots, \beta_n+1)} = \sum_{k=1}^{M} \frac{1}{z_{1(k)}^{\beta_1+1} \cdot z_{2(k)}^{\beta_2+1} \cdots z_{n(k)}^{\beta_n+1}}.$$

This expression is the sum of roots of system (5) to negative powers. The roots are not lying on coordinate planes.

**Theorem 2.** *For system (5) with polynomials $Q_j$ of the form (7) and $P_j$ of the form (8), for which*

$$l^1 + \ldots + l^n \leqslant \beta, \tag{10}$$

*where $l^j = (l_1^j, \ldots, l_n^j)$ and $l_i^j$ is the degree of polynomial $P_i$ with respect to variable $z_j$; $i, j = 1, \ldots, n$, the relation*

$$J_\beta = (-1)^n \sigma_{\beta+I},$$

*holds (multi-index $\alpha \leqslant \beta$, if this inequality is true for all coordinates).*

*Proof.* We perform the substitution of variables $z_j = \frac{1}{w_j}$, $j = 1, 2, \ldots, n$ in integral $J_\beta$. With this substitution the cycle $\gamma(r)$ is transformed to the cycle

$$(-1)^n \gamma\left(\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n}\right) = (-1)^n \gamma(R_1, R_2, \ldots, R_n).$$

Let us denote multi-index $\beta^j + s_j e^j$ as $\gamma^j$, $j = 1, 2, \ldots, n$. Then

$$\frac{df_j\left(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n}\right)}{f_j\left(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n}\right)} = \frac{d\widetilde{f}_j(w)}{\widetilde{f}_j(w)} - \sum_{k=1}^{n} \gamma_k^j \cdot \frac{dw_k}{w_k} - \sum_{k=1}^{n} \frac{1}{w_k^2} \cdot (P_j)'_{(z_k)} dw_k.$$

Therefore

$$J_\beta = \frac{(-1)^n}{(2\pi i)^n} \int_{\gamma(R)} w^{\beta+I} \left(\frac{d\widetilde{f}_1(w)}{\widetilde{f}_1(w)} - \sum_{k=1}^{n} \gamma_k^1 \cdot \frac{dw_k}{w_k} - \sum_{k=1}^{n} \frac{1}{w_k^2} \cdot (P_1)'_{(z_k)} dw_k\right) \wedge \ldots$$

$$\ldots \wedge \left( \frac{d\widetilde{f}_n(w)}{\widetilde{f}_n(w)} - \sum_{k=1}^{n} \gamma_k^n \cdot \frac{dw_k}{w_k} - \sum_{k=1}^{n} \frac{1}{w_k^2} \cdot (P_n)'_{(z_k)} dw_k \right).$$

We can easily show that all integrals of the form

$$\int_{\gamma(R)} w^{\beta+I} \frac{d\widetilde{f}_1(w)}{\widetilde{f}_1(w)} \wedge \ldots \wedge \frac{d\widetilde{f}_l(w)}{\widetilde{f}_l(w)} \wedge \frac{dw_{j_1}}{w_{j_1}} \wedge \ldots \wedge \frac{dw_{j_{n-l}}}{w_{j_{n-l}}} \tag{11}$$

not containing

$$\sum_{k=1}^{n} \frac{1}{w_k^2} \cdot (P_l)'_{(z_k)} dw_k$$

vanish if $0 \leqslant l < n$ and $R_j$ are sufficiently large.

In a similar way we can prove that if integrand expression contains the differentials $dP_j$ and $\dfrac{dw_k}{w_k}$ then these integrals also vanish.

Then we show that all integrals of the form

$$\int_{\gamma(R)} w^{\beta+I} \frac{d\widetilde{f}_1(w)}{\widetilde{f}_1(w)} \wedge \ldots \wedge \frac{d\widetilde{f}_l(w)}{\widetilde{f}_l(w)} \wedge dP_{l+1}\left(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n}\right) \wedge \ldots \wedge dP_n\left(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_n}\right) \tag{12}$$

with condition (10) vanish if $0 \leqslant l < n$ and $R_j$ are sufficiently large.

Thus, we have

$$J_\beta = \frac{(-1)^n}{(2\pi i)^n} \int_{\gamma(R)} w^{\beta+I} \frac{d\widetilde{f}_1(w)}{\widetilde{f}_1(w)} \wedge \ldots \wedge \frac{d\widetilde{f}_n(w)}{\widetilde{f}_n(w)}.$$

According to the Yuzhakov Theorem on Logarithmic Residue the last integral is equal the sum of values of holomorphic function $w^{\beta+I}$ at all roots of system (9). However, the value of function $w^{\beta+I}$ at the root of system (9), lying on coordinate plane, is equal to zero.

Therefore, we obtain

$$J_\beta = (-1)^n \sigma_{\beta+I}.$$

□

Let us extend our consideration. Let us assume that functions $f_j$ have the form

$$f_j(z) = \frac{f_j^{(1)}(z)}{f_j^{(2)}(z)}, \quad j = 1, 2, \ldots, n, \tag{13}$$

where $f_j^{(1)}(z)$ and $f_j^{(2)}(z)$ are entire functions in $\mathbb{C}^n$ of finite order of growth. They are represented by infinite product (uniformly converging in $\mathbb{C}^n$)

$$f_j^{(1)}(z) = \prod_{s=1}^{\infty} f_{j_s}^{(1)}(z), \quad f_j^{(2)}(z) = \prod_{s=1}^{\infty} f_{j_s}^{(2)}(z).$$

Moreover, each factor has the form $(z^{\beta^{j_s}} + Q_{j_s}(z))e^{P_{j_s}(z)}$. Polynomials $Q_{j_s}(z)$ and $P_{j_s}(z)$ are of the form (7), (8) and degrees of all polynomials $\deg P_{j_s} \leqslant \rho$, $j = 1, 2, \ldots, n$, $s = 1, 2, \ldots, \infty$.

Thus $f_j^{(1)}(z)$ и $f_j^{(2)}(z)$ are entire functions with finite order of growth not greater than $\rho$.

For all set of indexes $j_1, \ldots, j_n$, where $j_1, \ldots, j_n \in \mathbb{N}$, and each set of numbers $i_1, \ldots, i_n$, where $i_1, \ldots, i_n$ are equal to 1 or 2, systems of non-linear algebraic equations

$$f_{1j_1}^{(i_1)}(z) = 0, \quad f_{2j_2}^{(i_2)}(z) = 0, \ \ldots, \ f_{nj_n}^{(i_n)}(z) = 0, \tag{14}$$

have (according to Lemma 1) finite number of roots not lying on coordinate planes.

Number of roots of such system is not more than countable set. Let us denote the roots as $z_{(1)},\ z_{(2)},\dots,z_{(l)},\dots$.

Let us introduce the following expression

$$\sigma_{\beta+I}=\sum_{l=1}^{\infty}\frac{\varepsilon_l}{z_{1(l)}^{\beta_1+1}\cdot z_{2(l)}^{\beta_2+1}\cdots z_{n(l)}^{\beta_n+1}}.$$

Here $\beta_1,\dots,\beta_n$ are nonnegative integer numbers and the sign of $\varepsilon_l$ is equal to $+1$ if the system of the form (14), which root is $z_{(l)}$, contains even number of functions $f_{j_s}^{(2)}$; and the sign of $\varepsilon_l$ is equal to $-1$, if the system of the form (14), which root is $z_{(l)}$, contains odd number of functions $f_{j_s}^{(2)}$.

For system (5), which consists of functions of the form (13), the points $z_{(l)}$ are roots or singular points (poles). All functions $f_j$ are analytic in some neighborhood of 0.

Let us introduce multi-undex $l^j=(l_1^j,\dots,l_n^j)$, where $l_i^j$ is the maximum degree of polynomial $P_i$ with respect to variable $z_j$; $i,j=1,\dots,n$ contained in decomposition of $f_i$ (multi-index $\alpha\leqslant\beta$ if this inequality valid fore all coordinates).

**Theorem 3.** *Let us assume that the degrees of all polynomials $P_j$ used in decomposition of functions of the form (13) in system (5) are bounded by number $\rho$ and inequality*

$$l^1+\ldots+l^n\leqslant\beta,$$

*holds. Then the following relations*

$$J_\beta=(-1)^n\sigma_{\beta+I}$$

*are valid.*

The proof of this theorem immediately follows from Theorem 2.

# 4. Model of Zel'dovich–Semenov

We show that the considered methods of complex analysis can be useful in the study of the equations of chemical kinetics.

Consider the model of Zel'dovich-Semenov ideal mixing reactor (see. [9, Ch. 2, Eq. (2.2.1)]. It has the form

$$\begin{cases}(1-x)e^{\frac{y}{(1+\beta y)}}-\dfrac{x}{Da}=\dfrac{dx}{d\tau},\\(1-x)e^{\frac{y}{(1+\beta y)}}-\dfrac{y}{Se}=\gamma\dfrac{dy}{d\tau},\end{cases}$$

where $\beta,\ D,\ a,\ S,\ e$ are positive parameters.

Denote $Da=a$, $Se=b$. Stationary states of the system satisfy the equations

$$\begin{cases}(1-x)e^{\frac{y}{(1+\beta y)}}-\dfrac{x}{a}=0,\\(1-x)e^{\frac{y}{(1+\beta y)}}-\dfrac{y}{b}=0.\end{cases}\tag{15}$$

In [9, гл.2] qualitative study of the system conducted(15). We consider here a quantitative study.

From the Equations (15), we obtain that $x=\dfrac{a}{b}y$. Substituting this expression into the first equation, we have

$$e^{\frac{y}{1+\beta y}}=\frac{y}{b-ay}.$$

We make the substitution

$$\frac{y}{b - ay} = \frac{z}{b}, \tag{16}$$

then $y = \dfrac{zb}{b + az}$. Hence we have

$$e^{\frac{z}{1+z(\beta+a/b)}} = \frac{z}{b}. \tag{17}$$

We introduce the notation

$$\frac{1}{b} = \gamma, \quad \beta + \frac{a}{b} = \alpha,$$

i.e. $b = \dfrac{1}{\gamma}$, $\quad a = (\alpha - \beta)b$. Then from (17) we obtain the equation

$$e^{\frac{z}{1+\alpha z}} = \gamma z. \tag{18}$$

First examine the function

$$\varphi(z) = \frac{1}{z} \cdot e^{\frac{z}{1+\alpha z}}$$

for positive $z$. Find the derivative

$$\varphi'(z) = e^{\frac{z}{1+\alpha z}} \cdot \frac{-\alpha^2 z^2 - z(2\alpha - 1) - 1}{z^2(1 + \alpha z)^2}.$$

Investigate quadratic trinomial in the numerator of the fraction on the mark. Obtain that its discriminant $D = 1 - 4\alpha$, then at $0 < \alpha < 1/4$ derivative $\varphi'(z)$ has two roots $z_1 < z_2$, and at $\alpha > 1/4$ is one root. Exploring the position of the vertex of the parabola, we obtain that for $\alpha < 1/2$ it is positive, and for $\alpha > 1/2$ it is negative.

Therefore, if the derivative has two roots, they are both positive. In this case, the smaller root $z_1$ is a minimum point, and the larger root $z_2$ is a maximum point.

Asymptotes of the functions $\varphi(z)$ are: $z = 0$ is vertical asymptote ($\varphi(z) \to +\infty$ as $z \to +0$), and the axis $OZ$ is the horizontal asymptote ($\varphi(z) \to +0$ as $z \to +\infty$).

Consider the equation

$$\varphi(z) = \gamma, \tag{19}$$

equivalent to Equation (17).

From the previous studies, we obtain that Equation (19) at $0 < \alpha < 1/4$ has three roots at $\varphi(z_1) < \gamma < \varphi(z_2)$. And if $0 < \alpha < 1/4$, Equation (19) has one root, when either $z > \varphi(z_2)$, either $z < \varphi(z_1)$.

At $\alpha > 1/4$ Equation (19) has one root for all $\gamma$, since the function $\varphi$ is strictly decreasing. Calculating $z_1$ and $z_2$ at $\alpha < 1/4$, we obtain

$$z_1 = \frac{1 - 2\alpha - \sqrt{D}}{2\alpha^2}, \quad z_1 = \frac{1 - 2\alpha + \sqrt{D}}{2\alpha^2}, \quad D = 1 - 4\alpha.$$

Then

$$\varphi(z_1) = e^{\frac{1-\sqrt{D}}{2\alpha^2}} \cdot \frac{2\alpha^2}{1 - 2\alpha - \sqrt{D}},$$

and

$$\varphi(z_2) = e^{\frac{1+\sqrt{D}}{2\alpha^2}} \cdot \frac{2\alpha^2}{1 - 2\alpha + \sqrt{D}}.$$

**Proposition 1.** *Let $D = 1 - 4\alpha > 0$. Equation (19) has three positive roots at*

$$e^{\frac{1-\sqrt{D}}{2\alpha^2}} \cdot \frac{2\alpha^2}{1-2\alpha-\sqrt{D}} < \gamma < e^{\frac{1+\sqrt{D}}{2\alpha^2}} \cdot \frac{2\alpha^2}{1-2\alpha+\sqrt{D}},$$

*one root if either*

$$\gamma > e^{\frac{1+\sqrt{D}}{2\alpha^2}} \cdot \frac{2\alpha^2}{1-2\alpha+\sqrt{D}},$$

*either*

$$\gamma < e^{\frac{1-\sqrt{D}}{2\alpha^2}} \cdot \frac{2\alpha^2}{1-2\alpha-\sqrt{D}}.$$

*If $D = 1 - 4\alpha < 0$, Equation (19) has only one positive root.*

Returning to the variables $a,\ b, \beta$ we obtain

**Corollary 2.** *If $D' = \beta + \dfrac{a}{b} < 1/4$, then Equation (17) has three positive roots at*

$$e^{\frac{-1+\sqrt{D'}}{2(\beta+a/b)^2}} \cdot \frac{1-2(\beta+a/b)-\sqrt{D'}}{2(\beta+a/b)^2} > b > e^{\frac{-1-\sqrt{D'}}{2(\beta+a/b)^2}} \cdot \frac{1-2(\beta+a/b)+\sqrt{D'}}{2(\beta+a/b)^2},$$

*has one positive root, if either*

$$e^{\frac{-1+\sqrt{D'}}{2(\beta+a/b)^2}} \cdot \frac{1-2(\beta+a/b)-\sqrt{D'}}{2(\beta+a/b)^2} < b,$$

*either*

$$b < e^{\frac{-1-\sqrt{D'}}{2(\beta+a/b)^2}} \cdot \frac{1-2(\beta+a/b)+\sqrt{D'}}{2(\beta+a/b)^2}.$$

*At $\beta + \dfrac{a}{b} > 1/4$ Equation (17) has one positive root.*

Thus, the system (15) has no more than three roots with positive coordinates.
Let us consider how the system (15) has complex roots.
Solving it by making the change $t = \dfrac{y}{1+\beta y}$ (i.e. $y = \dfrac{t}{1-\beta t}$), we get

$$\begin{cases} \left(\dfrac{t}{b(1-\beta t)} - \dfrac{1}{a}\right) e^t + \dfrac{t}{ab(1-\beta t)} = 0, \\ x = 1 - \dfrac{t}{b(1-\beta t)} e^{-t}. \end{cases}$$

Hence

$$(at - b(1-\beta t))e^t + t = 0. \tag{20}$$

Denote by

$$\psi(t) = (at - b(1-\beta t))e^t + t.$$

Recall Hadamard theorem for functions of finite order of growth (see, for example, [8]).

**Definition 1.** *Expressions $E(u, 0) = 1 - u$,*

$$E(u, p) = (1-u)e^{u+\frac{u^2}{2}+\ldots+\frac{u^p}{p}};$$

$p = 1, 2, \ldots$ *are called primary factors.*

If the function $f(z)$ in the complex plane has a finite order of growth, then there is not depending on $n$ an integer $p \leqslant \rho$ that the product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right) \tag{21}$$

converges for all values of $z$, if the series converges

$$\sum \left(\frac{r}{r_n}\right)^{p+1} \tag{22}$$

where $r_1, r_2, \ldots$ are modules zeros of function $f(z)$, and this series converges for all values of $r$, if $p+1 \geqslant \rho$.

**Definition 2.** *Product (21) with the least of the integers p for which the series converges is called the canonical product, constructed from the zeros of $f(z)$, and is the smallest p is called its genus.*

**Theorem 4** (Hadamard)**.** *If a function $f(z)$ is entire of order $\rho$ with zeros $z_1, z_2, \ldots$, what is more $f(0) \neq 0$, then*

$$f(z) = e^{Q(z)} P(z), \tag{23}$$

*where $P(z)$ is canonical product constructed from the zeros of $f(z)$, and $Q(z)$ is polynomial of degree not higher than $\rho$ (see, for example, [8]).*

Function $\psi(t)$ is a entire function of the first order and exponential type 1. Let function $\psi(t)$ have a finite number of zeros in $\mathbb{C}$. Then by Hadamard's theorem it has the form

$$\psi(t) = e^t \cdot P_n(t),$$

where a polynomial

$$P_n(t) = \left(1 - \frac{t}{t_1}\right) \cdots \left(1 - \frac{t}{t_n}\right),$$

and $t_1,\ t_2,\ \ldots,\ t_n$ are zeros of function $\psi(t)$.

Then

$$(at - b(1-\beta t))e^t + t = e^t \cdot P_n(t).$$

Hence

$$e^t = \frac{-t}{at - b(1-\beta t) - P_n(t)},$$

which is impossible since the right is a rational function.

Thus the number of zeros of $\psi(t)$ is infinite. These zeros have no limit points in $\mathbb{C}$. If they are denoted by $t_1,\ \ldots, t_n, \ldots$, their modules $|t_n| \to \infty$ as $n \to \infty$.

Denote by $(x_n, y_n)$ $(n = 1, \ldots)$ are roots of the system (15). Since $y = \dfrac{t}{1-\beta t}$, then $y_n \to -\dfrac{1}{\beta}$ as $n \to \infty$. Since $\psi(t_n) = 0$, then

$$e^{t_n} = -\frac{t_n}{at_n - b(1-\beta t_n)},$$

hence $e^{t_n} \to -\dfrac{1}{a+b\beta}$.

Since $x = 1 - \dfrac{t}{b(1-\beta t)} e^{-t}$, then $x_n \to -\dfrac{a}{b\beta}$.

**Proposition 2.** *System (15) has an infinite number of complex roots* $(x_n, y_n) \in \mathbb{C}^2$*,* $n = 1, \ldots$*. there is a limit to this sequence of complex zeros when* $n \to \infty$ *and is equal to* $-\left(\frac{a}{b\beta}, \frac{1}{\beta}\right)$.

Let us consider the order of convergence of zeros $y_n$. Since the function $\psi(t)$ is a first order, then (see, for example, [8]) $\sum\limits_{n=1}^{\infty} \frac{1}{|t_n|^{1+\varepsilon}} < \infty$ for all $\varepsilon > 0$.

Hence we obtain

**Corollary 3.** *Series*

$$\sum_{n=1}^{\infty} \left| \frac{1}{y_n} + \beta \right|^{1+\varepsilon} < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

*This work was supported by the Russian Foundation for Basic Research, Grants 12-01-00007.*

# References

[1] L.A.Aizenberg, On a formula of the gereralized multidimensional logarithmic residue and the solution of system of nonlinear equations, *Sov. Math. Doc.*, **18**(1977), 691–695.

[2] L.A.Aizenberg, A.P.Yuzhakov, Integral representations and residues in multidimensional complex analysis, Trans. Amer. Math. Monographs, AMS, Providence, RI, 1983.

[3] A.K.Tsikh, Multidimensional residues and their applications. Trans. Amer. Math. Monographs, AMS, Providence, RI, 1992.

[4] V.Bykov, A.Kytmanov, M.Lazman, M.Passare, (ed), Elimination Methods in Polynomial Computer Algebra, Math. and Appl., Kluwer Acad. Publ., Dordreht, Boston, London, 1998.

[5] A.M.Kytmanov, Z.E.Potapova, Formulas for determining power sums of roots of systems of meromorphic functions, *Izvestiya VUZ. Matematika,* **49**(2005), no. 8, 36–45 (in Russian).

[6] A.M.Kytmanov, E.K.Myshkina, Evaluation of power sums of roots for systems of non-algebraic equations in $\mathbb{C}^n$, *Russ. Math.,* **57**(2013), no. 12, 31–43.

[7] A.A.Kytmanov, A.M.Kytmanov, E.K.Myshkina, Finding residue integrals for systems of non-algebraic equations in $\mathbb{C}^n$, *Journal of Symbolic Computation,* **66**(2015), 98–110.

[8] E.C.Titchmarsh, The Theory of Functions, Oxford Univerity Press, 1939.

[9] V.I.Bykov, S.B.Tsybenova, Non-linear models of chemical kinetics, KRASAND, Moscow, 2011 (in Russian).

# О некоторых системах неалгебраических уравнений в $\mathbb{C}^n$

**Ольга В. Ходос**

*Рассмотрен метод нахождения вычетных интегралов для систем неалгебраических уравнений, состоящих из целых функций. Такие интегралы связаны со степенными суммами корней системы уравнений. Предложенный подход может быть использован для развития метода исключения неизвестных из систем неалгебраических уравнений. Показано, что полученные результаты могут быть использованы для исследования одной модели химической кинетики.*

*Ключевые слова: неалгебраические системы уравнений, вычетный интеграл, степенные суммы.*