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Wave Propagation in a Blocky-layered Medium with Thin Interlayers

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Abstract. In this paper, a mathematical model of a blocky-layered medium is studied. Deformable elastic blocks and thin elastic and viscoelastic interlayers are considered. Viscoelasticity is taken into account to describe wave attenuation. The wave fields in a medium described by the proposed simplified interlayer model are compared to wave fields which were obtained using the equations of the dynamic elasticity theory for interlayers. The developed computational technology is verified for compatibility with the experimental data.

Keywords: blocky-layered media, thin interlayer, viscoelastic interlayer.

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Introduction

The concept of a blocky structure of rock masses was proposed by M. A. Sadovskii [1, 2]. According to this concept, the geological medium can be represented as a hierarchical structure consisting of blocks of different scales nested inside each other. The characteristic sizes of blocks may vary from several meters to tens of kilometers. In a medium where the interlayers are more compliant than the blocks, pendulum waves can be observed. Pendulum waves in blocky media is well studied in both theoretical and experimental aspects. When the deformations arise mainly in the interlayers, due to their high compliance, the blocks can be considered as rigid bodies. Discrete periodic models with rigid blocks connected to each other by elastic springs were represented in [3–5]. A similar but more complicated mathematical model that takes into account the elasticity of blocks was considered in [6]. The equations of this model are written relative to the central points of the blocks, and the accelerations of these points depend on the elastic moduli of both the blocks and the interlayers. Wave attenuation in blocky media may occur due to the viscoelasticity of the interlayer material. The behavior of a discrete-periodic medium with elastic blocks and viscoelastic interlayers is quite consistent with the experimental data [6].

A more complicated approach involves dynamic elasticity equations to describe deformations of blocks. Blocky-layered media with sufficiently large number of blocks can be represented as

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Cosserat continuum. The analysis of wave fields propagating in blocky-layered media and the Cosserat continuum was carried out in [7,8].

In this paper, we study two-dimensional blocky-layered media with elastic blocks and thin elastic and viscoelastic interlayers. A system of ordinary differential equations is used to describe dynamics of interlayers, while the equations of the dynamic elasticity theory in partial derivatives are used for blocks. A more consistent approach supposes to apply equations of elasticity theory for both blocks and interlayers. However, this method computationally seems to be more difficult, in particular due to different restrictions on the time step in blocks and interlayers. The proposed simplified model of a blocky-layered medium retains thermodynamical compatibility inherent in equations of elasticity theory.

We compare the numerical solutions obtained by the interlayer model described by the equations of elasticity theory and the proposed simplified model. It turns out that in a medium with interlayers and blocks of the same material, non-physical reflections of waves occur near the boundaries of the blocks, which indicates defectiveness of the simplified model. The numerical results for a blocky medium with thin viscoelastic interlayers are in agreement with the experimental data published in the work [6].

1. Mathematical model of a blocky-layered medium

A two-dimensional problem of the dynamics of a blocky-layered medium consisting of rectangular blocks is considered. Motion of each block complies with the system of equations of a homogeneous isotropic elastic medium:

$$\begin{gathered}
\rho\,\frac{\partial v_1}{\partial t}=\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{12}}{\partial x_2},\qquad \rho\,\frac{\partial v_2}{\partial t}=\frac{\partial \sigma_{12}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2},\\
\frac{\partial \sigma_{11}}{\partial t}=(\lambda+2\mu)\frac{\partial v_1}{\partial x_1}+\lambda\frac{\partial v_2}{\partial x_2},\qquad \frac{\partial \sigma_{22}}{\partial t}=\lambda\frac{\partial v_1}{\partial x_1}+(\lambda+2\mu)\frac{\partial v_2}{\partial x_2},\\
\frac{\partial \sigma_{12}}{\partial t}=\mu\Big(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\Big).
\end{gathered}\tag{1}$$

The equations of longitudinal and transverse motions in the elastic interlayer between adjacent blocks in the x_1 direction are written as follows:

$$\begin{gathered}
\rho'\,\frac{d}{dt}\frac{v_1^+ + v_1^-}{2}=\frac{\sigma_{11}^+ - \sigma_{11}^-}{\delta_1},\qquad \rho'\,\frac{d}{dt}\frac{v_2^+ + v_2^-}{2}=\frac{\sigma_{12}^+ - \sigma_{12}^-}{\delta_1},\\
\frac{d}{dt}\frac{\sigma_{11}^+ + \sigma_{11}^-}{2}=(\lambda'+2\mu')\frac{v_1^+ - v_1^-}{\delta_1},\qquad \frac{d}{dt}\frac{\sigma_{12}^+ + \sigma_{12}^-}{2}=\mu'\,\frac{v_2^+ - v_2^-}{\delta_1},
\end{gathered}\tag{2}$$

similarly, along the x_2 axis:

$$\begin{gathered}
\rho'\,\frac{d}{dt}\frac{v_2^+ + v_2^-}{2}=\frac{\sigma_{22}^+ - \sigma_{22}^-}{\delta_2},\qquad \rho'\,\frac{d}{dt}\frac{v_1^+ + v_1^-}{2}=\frac{\sigma_{12}^+ - \sigma_{12}^-}{\delta_2},\\
\frac{d}{dt}\frac{\sigma_{22}^+ + \sigma_{22}^-}{2}=(\lambda'+2\mu')\frac{v_2^+ - v_2^-}{\delta_2},\qquad \frac{d}{dt}\frac{\sigma_{12}^+ + \sigma_{12}^-}{2}=\mu'\,\frac{v_1^+ - v_1^-}{\delta_2}.
\end{gathered}\tag{3}$$

Here v_1, v_2 are components of the displacement velocity vector, σ_{11}, σ_{22}, σ_{12} are components of the stress tensor, $\lambda=\rho(c_p^2-2c_s^2)$, $\mu=\rho c_s^2$ are Lame parameters, ρ is density, c_p, c_s are the velocities of longitudinal and transverse elastic waves, respectively, strokes indicate constants for interlayers. The interlayer thickness in both directions assumed to be the same $\delta=\delta_1=\delta_2$.

Signs «+» and «–» are represent the values on the right and left boundaries of the interlayer, respectively.

The system (1)–(3) is thermodynamically compatible. The law of conservation of energy can be written down as the sum of the kinetic and potential energies of all blocks and interlayers, equal to the integral of flux of the Umov–Poynting vector by time and across the boundary of the block array consisting of $n_1 \times n_2$ blocks [7]:

$$
\begin{aligned}
&\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\int_0^{h_1}\int_0^{h_2}\Big(\frac{\rho}{2}\vec{v}^{k_1,k_2}(t,x_1,x_2)^2 + W^{k_1,k_2}(t,x_1,x_2)\Big)dx_1dx_2+\\
&+\delta_1\sum_{k_1=1}^{n_1-1}\sum_{k_2=1}^{n_2}\int_0^{h_2}\Big(\frac{\rho'}{2}\Big[\frac{\vec{v}^{k_1+1,k_2}(t,0,x_2)+\vec{v}^{k_1,k_2}(t,h_1,x_2)}{2}\Big]^2+\\
&+\frac{1}{2\rho' c'^2_p}\Big[\frac{\sigma_{11}^{k_1+1,k_2}(t,0,x_2)+\sigma_{11}^{k_1,k_2}(t,h_1,x_2)}{2}\Big]^2+\\
&+\frac{1}{2\rho' c'^2_s}\Big[\frac{\sigma_{12}^{k_1+1,k_2}(t,0,x_2)+\sigma_{12}^{k_1,k_2}(t,h_1,x_2)}{2}\Big]^2\Big)\,dx_2+\\
&+\delta_2\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2-1}\int_0^{h_1}\Big(\frac{\rho'}{2}\Big[\frac{\vec{v}^{k_1,k_2+1}(t,x_1,0)+\vec{v}^{k_1,k_2}(t,x_1,h_2)}{2}\Big]^2+\\
&+\frac{1}{2\rho' c'^2_p}\Big[\frac{\sigma_{22}^{k_1,k_2+1}(t,x_1,0)+\sigma_{22}^{k_1,k_2}(t,x_1,h_2)}{2}\Big]^2+\\
&+\frac{1}{2\rho' c'^2_s}\Big[\frac{\sigma_{12}^{k_1,k_2+1}(t,x_1,0)+\sigma_{12}^{k_1,k_2}(t,x_1,h_2)}{2}\Big]^2\Big)\,dx_1=\\
&=\sum_{k_2=1}^{n_2}\int_0^{t}\int_0^{h_2}\Big(p_1^{n_1,\,k_2}(t,h_1,x_2)-p_1^{0,\,k_2}(t,0,x_2)\Big)dx_2dt+\\
&+\sum_{k_1=1}^{n_1}\int_0^{t}\int_0^{h_1}\Big(p_2^{k_1,\,n_2}(t,x_1,h_2)-p_2^{k_1,\,0}(t,x_1,0)\Big)dx_1dt.
\end{aligned}
\tag{4}
$$

Here, $\vec{v} = (v_1, v_2)$ is the velocity vector, $p_1 = \sigma_{11}v_1+\sigma_{12}v_2$, $p_2 = \sigma_{22}v_2+\sigma_{12}v_1$ are the projections of the power flux vector, W is the elastic potential:

$$
W = \frac{(\sigma_{11}+\sigma_{22})^2}{8(\lambda+\mu)} + \frac{(\sigma_{11}-\sigma_{22})^2+4\sigma_{12}}{8\mu}. \tag{5}
$$

Thermodynamic compatibility guarantees the well-posedness of the initial-boundary value problem with the dissipative boundary conditions, under which the right-hand side of (4) is non-negative.

We consider a boundary value problem for a blocky-layered massif with fixed boundaries ($v_1 = v_2 = 0$ along the boundaries). The numerical solution is calculated in the region $\Omega = [0, L_1] \times [0, L_1]$ with a uniform grid of $N_1 \times N_2$ nodes. At the boundary $x_1 = 0$, at point $x_2 = x_{imp}$ the pressure pulse is $\sigma_{11}(t, x_{imp}) = p(t)$.

The numerical algorithm for solving equations in blocks is based on the two-cyclic splitting method with the respect to spatial coordinates. This method allows to achieve the second order of convergence when splitted one-dimensional problems are solved by finite-difference schemes of at least second order [9]. Godunov scheme with limiting reconstruction of Riemann invariants is used to solve one dimensional problems [10]. The reconstruction procedure provides second-order approximation in monotonic sections of the solution.

Equations in the interlayers are solved using Ivanov dissipation-free scheme [11]. To eliminate artificial scheme dissipation, it is necessary to require that the sum of the values at the upper (indicated by the "hat" symbol) and lower time steps be equal to the sum of the values at the left and right boundaries of the grid cell:

$$v^+ + v^- = \hat{v} + v, \qquad \sigma^+ + \sigma^- = \hat{\sigma} + \sigma.$$

Based on this requirement, at the "predictor" stage of the scheme we obtain the system [13]:

$$\begin{gathered} I^+ = \rho c v^+ + \sigma^+, \quad I^- = \rho c v^- - \sigma^-, \\ v^+ - v^- = \frac{\delta}{\rho c \delta + \rho' c'^2 \tau}(I^+ - I^- - 2\sigma), \qquad \sigma^+ - \sigma^- = \frac{\rho' \delta}{\rho' \delta + \rho c \tau}(I^+ + I^- - 2\rho c v), \\ v^+ + v^- = \frac{I^+ + I^- - (\sigma^+ - \sigma^-)}{\rho c}, \qquad \sigma^+ + \sigma^- = I^+ - I^- - \rho c(v^+ - v^-). \end{gathered} \tag{6}$$

Here I^+ and I^- are Riemann invariants, calculated on the boundaries of neighboring blocks separated by an interlayer, τ is the time step and h is the space step. On the "corrector" stage we have a system:

$$\hat{v} = v + \frac{\tau}{\rho \delta}(\sigma^+ - \sigma^-), \qquad \hat{\sigma} = \sigma + \frac{\tau \rho c^2}{\delta}(v^+ - v^-). \tag{7}$$

This scheme can be written for independent subsystems for longitudinal and transverse waves propagating with velocities $c = c_p$ and $c = c_s$, respectively. It is necessary to allocate one-dimensional arrays in each direction for interlayers. That is, when solving splitted one-dimensional problems, stresses and strain rates in each interlayer are calculated in only one grid cell.

To account mechanical energy dissipation, we consider viscoelastic interlayers. Viscoelastic interlayers are described by the Poynting–Thomson model, also known as the standard linear solid (SLS) model. The rheological scheme of the model consists of elastic element b_0 connected in series with a parallel connection of a viscous element η and an elastic one b (Fig. 1). An array of SLS-mechanisms connected in parallel is called a generalized standard linear solid (GSLS). This rheological model is widely used in geophysics due to its ability to describe media with a nearly constant quality factor over a certain frequency range. The more mechanisms in the model, the more precisely constant quality factor can be approximated. Denoting s as the stress on elastic element b we can write down the system of equations for the viscoelastic interlayer in the following form:

$$\begin{gathered} \rho' \frac{d}{dt}\frac{v^+ + v^-}{2} = \frac{\sigma^+ - \sigma^-}{\delta_1}, \qquad \frac{1}{b_0}\frac{d}{dt}\frac{\sigma^+ + \sigma^-}{2} = \frac{v^+ - v^-}{\delta_1} - \frac{1}{\eta}\left(\frac{\sigma^+ + \sigma^-}{2} - \frac{s^+ + s^-}{2}\right), \\ \frac{1}{b}\frac{d}{dt}\frac{s^+ + s^-}{2} = \frac{1}{\eta}\left(\frac{\sigma^+ + \sigma^-}{2} - \frac{s^+ + s^-}{2}\right). \end{gathered} \tag{8}$$

For longitudinal waves, the coefficient b is equal to $\lambda + 2\mu$ and for transverse waves is equal to μ, and similarly, b_0 is equal to $\lambda_0 + 2\mu_0$ or μ_0.

This model can be rewritten in terms of the relaxation modulus and relaxation times of stress and strain. Relaxation times can be determined from the known quality factor using the τ-method [12]. Then we can recalculate the elastic moduli and viscosity coefficient.

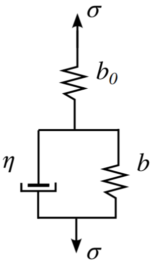

Fig. 1. Rheological scheme of the Poynting–Thomson model

Finite-difference scheme is constructed analogously to (6)–(7), but leads to a more cumbersome form:

$$v^+ - v^- = \frac{1}{\alpha}\big(\beta(I^+ - I^-) - 2\sigma - \frac{2b_0\tau}{2\eta + b\tau}s\big), \qquad \sigma^+ - \sigma^- = \frac{\rho'\delta}{\rho'\delta + \rho c\tau}\big(I^+ + I^- - 2\rho c v\big),$$

$$v^+ + v^- = \frac{I^+ + I^- - (\sigma^+ - \sigma^-)}{\rho c}, \quad \sigma^+ + \sigma^- = I^+ - I^- - \rho c(v^+ - v^-)\,,$$

$$s^+ + s^- = \frac{b\tau}{2\eta + b\tau}\big(I^+ - I^- - \rho c(v^+ - v^-)\big) + \frac{4\eta}{2\eta + b\tau}s\,,$$

$$\hat{v} = v + \tau\frac{\sigma^+ - \sigma^-}{\delta\rho'}\,, \quad \hat{\sigma} = \sigma + b_0\tau\frac{v^+ - v^-}{\delta} - \frac{b_0\tau}{\eta}\left(\frac{\sigma^+ + \sigma^-}{2} - \frac{s^+ + s^-}{2}\right),$$

$$\hat{s} = s + \frac{b\tau}{\eta}\left(\frac{\sigma^+ + \sigma^-}{2} - \frac{s^+ + s^-}{2}\right),$$

where

$$\alpha = \rho c + \frac{b_0\tau}{\delta} + \frac{b_0\tau\rho c}{2\eta}\left(1 - \frac{b\tau}{2\eta + b\tau}\right), \qquad \beta = 1 + \frac{b_0\tau}{2\eta}\left(1 - \frac{b\tau}{2\eta + b\tau}\right).$$

2. Results of computations

The computations below were performed on a multiprocessor system with cluster architecture. The software package was developed using the MPI library. Each MPI-process performs computations on each block, which consists of smaller blocks. One can specify different interlayer thicknesses for larger and smaller blocks, so that it is possible to simulate wave propagation in hierarchical blocky media.

Simplification of the interlayer model leads to certain inaccuracies. Let us evaluate the behavior of the wave field when propagating near the boundaries of blocks. We consider a medium consisting of four identical rectangular blocks with sides of 12 and 24 m, separated by interlayers with varying thickness. The first case considered concerns a medium with the blocks and interlayers of the same material with properties $\rho = \rho' = 2400$ kg/m^3, $c_p = c'_p = 4500$ m/s, $c_s = c'_s = 2700$ m/s. The load pressure at the upper boundary of the first block at point $x_{imp}=21$ m is $p(t) = p_0 H(t)$, where $H(t)$ is Heaviside function. The Fig. 2 shows snapshots of velocity fields obtained using interlayer model (2)–(3) calculated on uniform grid of $N_1 \times N_2 = 480 \times 960$ nodes (with $h = 0.5$ m). In the medium, where interlayers are modeled by the same equations as blocks, waves propagate like in a homogenous medium. As the thickness of the interlayers increases, partial reflections of waves from the vertical layer become more and more

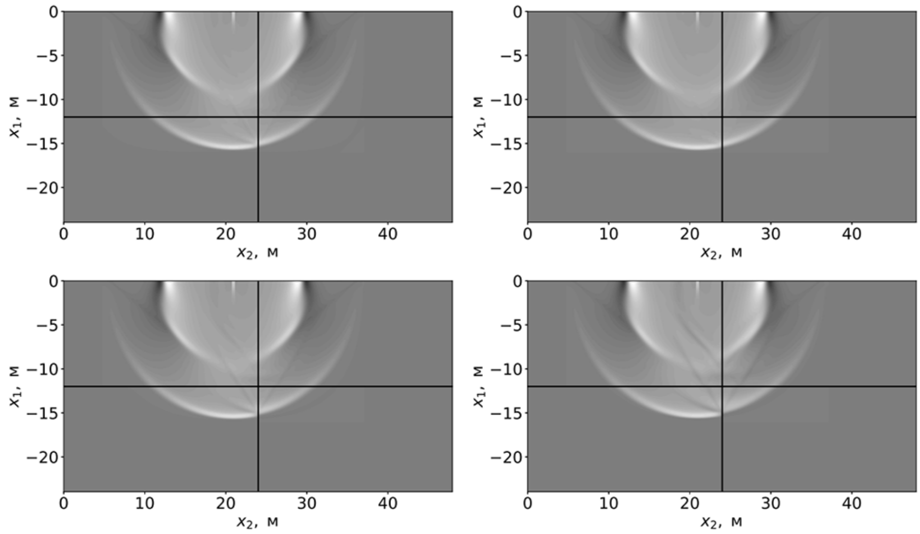

Fig. 2. Snapshots of the velocity v_1 in medium with blocks and interlayers made of the same material obtained using simplified intarlayer model (2)–(3), interlayer thicknesses are $\delta = 0.025$ m (upper left), 0.05 m (upper right), 0.1 m (bottom left), 0.2 m (bottom right)

sufficient. There are almost no reflections from the horizontal interlayer, since the wave passes through it almost perpendicularly.

Let us consider a medium of the same configuration but with a more compliant interlayer material: $\rho' = 2100$ kg/m^3, $c_p' = 2900$ m/s, $c_s' = 1700$ m/s. In this case no visual differences between the snapshots obtained by different interlayer models are observed (Fig. 3). To es-

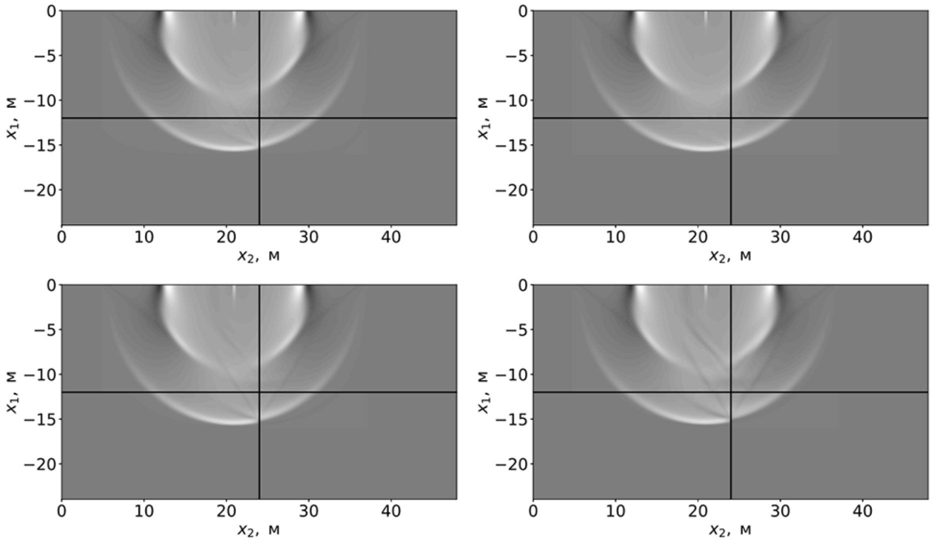

Fig. 3. Snapshots of the velocity v_1 in medium with compliant interlayers $\delta = 0.05$ m (left) and 0.2 m (right) thick for simplified interlayer model (2)–(3) (upper) and for interlayers described by elasticity theory equations (bottom)

timate the error of the numerical solution U obtained with the use of a simplified interlayer model (2)–(3), we compare it to a reference solution U_e calculated for interlayers described by elasticity theory equations. The relative error $err_2 = ||U - U_e||/||U_e||$ of the numerical solution $U = (v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{12})$ was calculated using a discrete equivalent of the norm of the space $L_\infty(0,T;\, L_2(\Omega))$:

$$||U|| = \sup_{0<t<T} \sqrt{\iint\limits_{\Omega} \left(\rho \frac{v_1^2 + v_2^2}{2} + W\right) dx_1\, dx_2},$$

where T is the time required for the longitudinal wave to reach the boundary of the computational domain Ω, W is the elastic potential (5). Also we use the norm

$$||U|| = \sup_{0<t<T} \max_{\Omega} |U|$$

to calculate relative error err_C.

Tab. 1 shows the relative errors depending on grid step h for a fixed interlayer thickness. The material of the blocks for all cases has parameters $\rho = 2400$ kg/m^3, $c_p = 4500$ m/s, $c_s = 2700$ m/s, the material of the interlayers varies. Tab. 2 shows the relative errors depending on interlayer

Table 1. The relative error depending on grid step at fixed interlayer thickness $\delta = 0.1$ m

Interlayer material parameters		$\rho' = \rho$, $c_p' = c_p$, $c_s' = c_s$		$\rho' = 2100$ kg/m^3, $c_p' = 2900$ m/s, $c_s' = 1700$ m/s		$\rho' = 1100$ kg/m^3, $c_p' = 1500$ m/s, $c_s' = 800$ m/s	
h, м	δ/h	err_2	err_C	err_2	err_C	err_2	err_C
0.1	1	0.0352	0.272	0.0303	0.123	0.0231	0.0715
0.05	2	0.0483	0.321	0.0286	0.144	0.0258	0.0839
0.025	4	0.0621	0.348	0.0357	0.146	0.0309	0.115
0.0125	8	0.0802	0.350	0.0503	0.153	0.0475	0.131

thickness with a fixed grid. With an increase in the ratio of the interlayer thickness to the grid

Table 2. The relative error depending on interlayer thickness at fixed grid $N_1 \times N_2 = 960 \times 1920$ ($h = 0.025$ m)

Interlayer material parameters		$\rho' = \rho$, $c_p' = c_p$, $c_s' = c_s$		$\rho' = 2100$ kg/m^3, $c_p' = 2900$ m/s, $c_s' = 1700$ m/s		$\rho' = 1100$ kg/m^3, $c_p' = 1500$ m/s, $c_s' = 800$ m/s	
δ, м	δ/h	err_2	err_C	err_2	err_C	err_2	err_C
0.025	1	0.0195	0.154	0.0177	0.0756	0.0117	0.0597
0.05	2	0.0362	0.250	0.0211	0.108	0.0197	0.0704
0.1	4	0.0621	0.348	0.0357	0.146	0.0309	0.115
0.2	8	0.1035	0.414	0.0719	0.161	0.0689	0.237

step δ/h, an increase in error is observed in all cases. It is noticeable that in media with more compliant interlayers the error is slightly lower. Therefore, the model with simplified equations for interlayers can be used to describe blocky media with sufficiently thin and compliant interlayers.

Fig. 4 shows the distribution of the error $|v_1 - v_{1e}|/|v_{1e}|$ in blocky media for layers of different thicknesses on a uniform grid $N_1 \times N_2 = 960 \times 1920$ ($h = 0.025$ m).

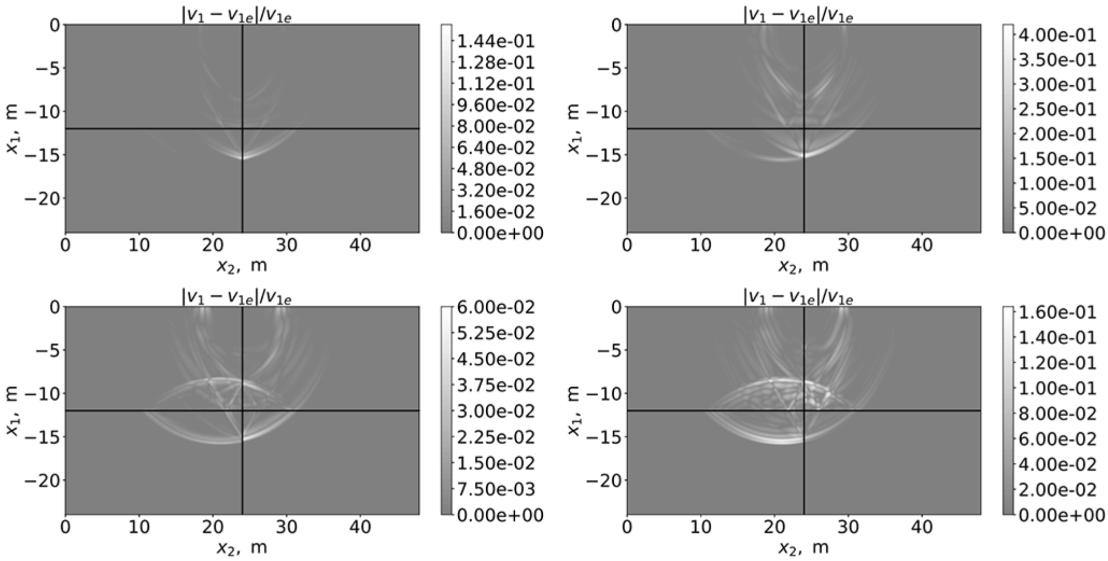

Fig. 4. The relative error in a blocky-layered medium with interlayers $\delta = 0.025$ m (left) and $\delta = 0.2$ m (right) thick, interlayer material with $\rho' = \rho$, $c_p' = c_p$, $c_s' = c_s$ (upper), a more compliant interlayer material $\rho' = 2100$ kg/m^3, $c_p' = 2900$ m/s, $c_s' = 1700$ m/s (bottom)

Verification of the mathematical model and computational technology was carried out according to experimental data published in paper [6]. In the experiments on a biaxial stand, a blocky-layered medium was simulated by an assembly of 36 blocks measuring $89 \times 125 \times 250$ mm, each made of plexiglass ($\rho = 2040$ kg/m^3, $c_p = 2670$ m/s). Blocks were separated by 5 mm thick rubber interlayers with shear moduli in directions x_1 and x_2 equal to $10^7/1.3$ Pa and $1.35 \cdot 10^7/1.3$ Pa, respectively.

It was assumed that the shear moduli of the interlayers correspond to the state of long-term deformation, when both elements of the rheological scheme are deformed (Fig. 1). The Poisson's ratio for all assembly materials was assumed to be 0.3. Fig. 5 shows the diagram of the numerical experiment. The rod striker generated elastic waves in contact with the surface of the block. At point x_{imp}, denoted by the red arrow in Fig. 6, the pulse impact $\sigma_{11}(t, x_{imp}) = p(t)$ with

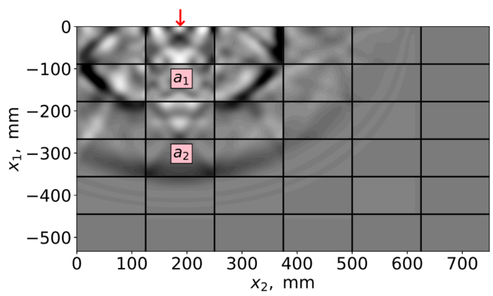

Fig. 5. The numerical experiment diagram. Accelerometers a_1 and a_2 are placed in the central points of the corresponding blocks

duration $T_{imp} = 0.2$ ms has the following form:

$$p(t) = \begin{cases} p_0 \sin(\pi t / T_{imp}), & 0 < t \leqslant T_{imp} \\ 0, & t > T_{imp}. \end{cases}$$

Accelerometers a_1 and a_2 were measuring accelerations $w_i = \partial v_i / \partial t$ for 5 ms in the central points of the corresponding blocks.

Figures 6–9 show the theoretical and experimental results from paper [6] in comparison with the numerical solution of (1) and (8). The experimental dependencies of acceleration on time are denoted by the blue dashed line, the red lines show accelerations calculated using the approach proposed in [6], the green curves correspond to the numerical solution for a medium with elastic blocks and viscoelastic interlayers. The parameters of the SLS were obtained using the τ-method [12] assuming that quality factor Q is nearly constant in the frequency range $[100, 5000]$ Hz. It was assumed that quality factors of the longitudinal and transverse waves are $Q_p = 20$ and $Q_s = 10$, respectively. The lack of data on the material of the interlayers leaves a certain amount of arbitrariness in the choice of parameters of the viscoelastic medium.

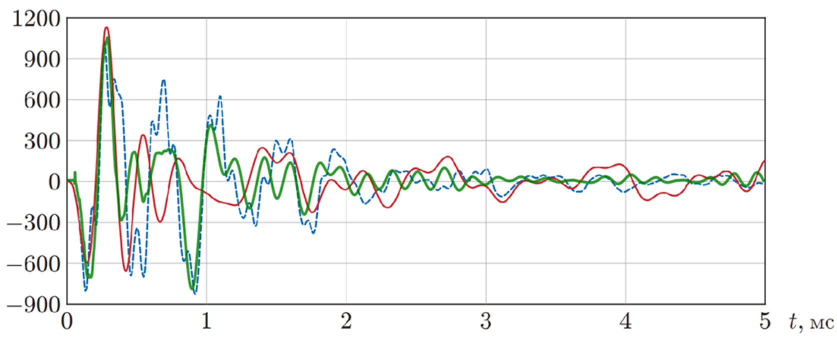

Fig. 6. Waveforms of acceleration w_1, measured in the centre of block a_1

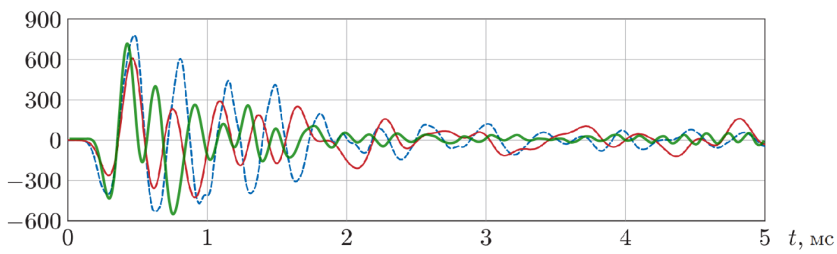

Fig. 7. Waveforms of acceleration w_1, measured in the centre of block a_2

The results of the numerical simulation are in good agreement with experimental data. The calculated acceleration waveform shown in Fig. 6 is almost identical to the experimental measurement. In Fig. 7 one can see the difference in phase, but the qualitative behaviour of the waves remains the same. A more observable difference can be noted in Fig. 8–9 where the experimental high-frequency oscillations with large amplitude were not detected numerically. Most likely, this is due to the fact that accelerations in the experiment were measured on the side surface of the block, while the two-dimensional problem supposes measurements "inside" the thickness of the

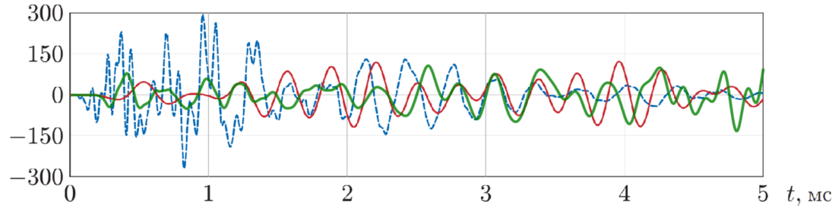

Fig. 8. Waveforms of acceleration w_2, measured in the centre of block a_1

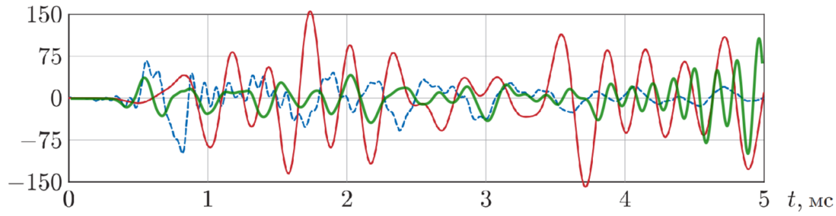

Fig. 9. Waveforms of acceleration w_2, measured in the centre of the block a_2

block. It would be more accurate to apply a three-dimensional model of a blocky-layered medium with the same location of accelerometers as in the real experiment.

Conclusions

The considered simplified interlayer model reliably describes wave processes in blocky-layered media. When blocks and interlayers are made of the same material, non-physical reflections occur and grow as the interlayers get thicker. The solutions for the simplified interlayers model and for the interlayers described by elasticity theory equations are compared. It is observed that the error of the numerical solution obtained by the simplified model increases with increasing ratio of the interlayer thickness to the grid step. The mathematical model was verified on the experimental data published in paper [6]. The presented computations show good agreement with the experimental measurements.

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Распространение волн в блочно-слоистой среде с тонкими прослойками

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Аннотация. Исследуется математическая модель блочно-слоистой среды с тонкими прослойками. Рассматриваются деформируемые упругие блоки и упругие прослойки. Для описания затухания волн учитывается вязкоупругость в прослойках. Проводится численное сравнение упрощённой модели прослоек с прослойками, описываемыми полными уравнениями теории упругости. Результаты численного моделирования сравниваются с экспериментальными данными.

Ключевые слова: блочно-слоистая среда, тонкая прослойка, вязкоупругая прослойка.