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An Identification Problem of Memory Function of a Medium and the Form of an Impulse Source

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In the paper the problem of identification of two functions, one of which is under the integral sign in a hyperbolic equation and represents the medium's memory, the other one defines the regular part of an impulse source is considered. As additional information Fourier image trace solution of a direct problem on the hyperplane $y = 0$ for two different values of transformation parameters is given. The estimate of stability of the solution of the inverse problem and the uniqueness theorems is proved.

Keywords: inverse problem, hyperbolic equation, stability, uniqueness.

1. Setting up the Problem

We consider the boundary value problem

$$u_{tt} - u_{xx} - u_{yy} - \int_0^t k(\tau)u_{xx}(x, y, t-\tau)d\tau = 0, \quad (x, y, t) \in R^2_+ \times R, \tag{1.1}$$

$$u|_{t<0} \equiv 0, \quad u_y|_{y=0} = a\delta(x)\delta^{'}(t) + f(t)\delta(x)\theta(t), \tag{1.2}$$

where $R^2_+ = \left\{(x, y) \in R^2 | y > 0\right\}$; $\theta(t) = 1,\ t \geqslant 0$; $\theta(t) = 0,\ t < 0$; $\delta(t) = (d/dt)\theta(t),\ \delta'(t) = (d^2/dt^2)\theta(t),\ a \neq 0$ is a real number.

For given functions $k(t),\ f(t)$ and a number a we call the „direct problem" the problem of finding the function $u(x, y, t)$, that satisfies (in a generalized meaning) the equations (1.1), (1.2).

We assume that a solution to this problem is defined on the boundary of the domain $R^2_+ \times R$

$$u|_{y=0} = F(x, t),\ (x, t) \in R^2_+. \tag{1.3}$$

The inverse problem is to in determine the two functions $k(t),\ f(t)$ and a number a for a given function $F(x, t)$.

Definition 1.1. *A pair of functions $k(t)$, $f(t)$ and a number a are calld the solution of a inverse problem (1.1)–(1.3) if the corresponding solution of the problems (1.1),(1.2) satisfies the equation (1.3).*

Regarding the problems devoted to determining a sub-integral function, belonging to hyperbolic equations, see works [1,2]. In the work [1] the problem of finding out the memory, belonging to a three-dimensional wave equation with delta function at the right side is investigated. Further, in work [2] this problem is generalized in the case of a hyperbolic equation

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of the second order with a constant main part and variable coefficients at minor derivatives. Similar problems with distributed sources of disturbance can be found in works [3,4]. In the article [5] the problem of determining a one-dimensional coefficient of wave distribution velocity and forms of impulse sources according to the information (1.3) is investigated. It turns out that to solve this problem it is sufficient to give the Fourier image of the function $F(x,t)$ in two different values of the transformation parameter. In this article we investigate the determination of two functions of one variable, one of which is under the integral sign similar to the method of the work [5].

Let $\tilde{u}(\lambda, y, t)$ be the Fourier transformation of the function $u(x, y, t)$ with respect to the variable x:

$$\tilde{u}(\lambda, y, t) = \frac{1}{\sqrt{2\pi}} \int\limits_R u(x, y, t) e^{i\lambda x} dx.$$

From the theory of hyperbolic equations, we know that for given functions $k(t)$ and $f(t)$ problem (1.1), (1.2) correctly defines the function $u(x, y, t)$, possessing the compact support at any finite t. In terms of the Fourier transform the problem (1.1)–(1.3) is reduced to the form

$$\tilde{u}_{tt} - \tilde{u}_{yy} + \lambda^2 \tilde{u} + \lambda^2 \int\limits_0^t k(\tau) \tilde{u}(\lambda, y, t-\tau) d\tau = 0, \quad (\lambda, y, t) \in R_+^2 \times R, \tag{1.4}$$

$$\tilde{u}|_{t<0} \equiv 0, \tilde{u}_y|_{y=0} = a\delta^{'}(t) + f(t)\theta(t), \tag{1.5}$$

$$\tilde{u}|_{y=0} = \tilde{F}(\lambda, t), \quad (\lambda, t) \in R_+^2. \tag{1.6}$$

The solution of the direct problem (1.4), (1.5) can be represented as

$$\tilde{u}(\lambda, y, t) = -a\delta(t-y) + \theta(t-y) v(\lambda, y, t). \tag{1.7}$$

It is easily seen from the relations (1.4), (1.5), that the function $v(\lambda, y, t)$ at a fixed λ in the domain $t > y > 0$ satisfies the equations

$$v_{tt} - v_{yy} + \lambda^2 v(\lambda, y, t) - \lambda^2 a k(t-y) + \lambda^2 \int\limits_0^{t-y} k(\tau) v(\lambda, y, t-\tau) d\tau = 0, \tag{1.8}$$

$$v|_{t=y+0} = \frac{\lambda^2 a}{2} y, \tag{1.9}$$

$$v_y|_{y=0} = f(t). \tag{1.10}$$

We observe that for the solvability of the inverse problem, as concluded from the presentation (1.7), the function $\tilde{F}(\lambda, t)$ must have the following structure:

$$\tilde{F}(\lambda, t) = -a\delta(t) + F_0(\lambda, t)\theta(t), \quad (\lambda, t) \in R_+^2, \tag{1.11}$$

where the function $F_0(\lambda, t)$ at the argument t satisfies some conditions of compatibility and smoothness, that will be addressed later. The formula (1.11) demonstrates that the inverse problem data make it possible to find a as a coefficient at the singular part of $\tilde{F}(\lambda, t)$ and

$a \neq 0$. Further we shall assume, that a is a known number. It follows from the formula (1.11) that the additional information (1.6) for the function v has the form

$$v|_{y=0} = F_0(\lambda, t), \quad (\lambda, t) \in R_+^2. \tag{1.12}$$

Now we narrow the data of the inverse problem, assuming that the function $F_0(\lambda, t)$ (consequently the function $\tilde{F}(\lambda, t)$) is known for only two values λ_1, λ_2 such that $\lambda_1^2 \neq \lambda_2^2$. Thus, the inverse problem (1.1)–(1.3) is brought to the problem of identifying the functions $k(t), f(t)$ from the relations (1.8)–(1.10) if the solution of the direct problem is known for $\lambda = \lambda_j, j = 1, 2$ and is given by the equation (1.12). It turns out that by these data the functions $k(t), f(t)$ are uniquely defined.

2. The Direct Problem

We analyze the properties of the solution to the direct problem (1.8)–(1.10).

Lemma 2.1. *Let $k(t) \in C[0,T]$, $f(t) \in C[0,T]$ at some $T > 0$. Then at each fixed value of the parameter λ the solution of the problems (1.8)–(1.10) for $(y,t) \in D_T$, $D_T = \{(y,t)| 0 \leqslant y \leqslant t \leqslant T - y\}$ belongs to the functional class $C^1(D_T)$ and for the solution the estimate*

$$\|v\|_{C^1(D(T)} \leqslant C \left(|a| + \|k(t)\|_{C[0,T]} + \|f\|_{C[0,T]}\right), \tag{2.1}$$

is valid, with C depending only on T, λ , $\|k(t)\|_{C[0,T]}$, a. Besides, the functions

$$\psi(\lambda_1, \lambda_2, t) = v_t(\lambda_1, 0, t) - v_t(\lambda_2, 0, t)$$

at any fixed λ_j, $j = 1, 2$ belong to the class $C^1[0,T]$.

Proof. Since

$$v_{tt} - v_{yy} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right)(v_t + v_y) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)(v_t - v_y),$$

it follows that for $(y,t) \in D_T$ from the relations (1.8)–(1.10) (by integrating along the corresponding characteristics of the derivative operators of the first order) yield the equations

$$(v_t + v_y)(\lambda, y, t) = \frac{\lambda^2 a}{2} +$$

$$+\lambda^2 \int\limits_y^{(y+t)/2} \left[ak(t + y - 2\xi) - v(\lambda, \xi, t + y - \xi) - \int\limits_0^{t+y-2\xi} k(\tau) v(\lambda, \xi, t + y - \xi - \tau) d\tau\right] d\xi, \tag{2.2}$$

$$v(\lambda, 0, t) = \frac{\lambda^2 a}{2} t - \int\limits_0^t f(\tau) d\tau +$$

$$+\lambda^2 \int\limits_0^t \int\limits_0^{\tau/2} \left[ak(\tau - 2\xi) - v(\lambda, \xi, \tau - \xi) - \int\limits_0^{\tau-2\xi} k(\alpha) v(\lambda, \xi, \tau - \xi - \alpha) d\alpha\right] d\xi d\tau, \tag{2.3}$$

$$(v_t - v_y)(\lambda, y, t) = \frac{\lambda^2 a}{2} + \lambda^2 a k(t - y) y - 2f(t - y) +$$

$$+\lambda^2 \int\limits_0^{(t-y)/2} \left[ak(t-y-2\xi) - v(\lambda,\xi,t-y-\xi) - \int\limits_0^{t-y-2\xi} k(\tau)v(\lambda,\xi,t-y-\xi-\tau)d\tau \right] d\xi -$$

$$-\lambda^2 \int\limits_0^{y} \left[v(\lambda,\xi,t-y+\xi) + \int\limits_0^{t-y} k(\tau)v(\lambda,\xi,t-y+\xi-\tau)d\tau \right] d\xi. \quad (2.4)$$

From (2.2), (2.4), we find

$$v_t(\lambda,y,t) = \frac{\lambda^2 a}{2} + \frac{\lambda^2 a}{2}k(t-y)y - f(t-y) +$$

$$+\frac{\lambda^2}{2}\int_0^{(t-y)/2} \left[ak(t-y-2\xi) - v(\lambda,\xi,t-y-\xi) - \int\limits_0^{t-y-2\xi} k(\tau)v(\lambda,\xi,t-y-\xi-\tau)d\tau \right] d\xi -$$

$$-\frac{\lambda^2}{2}\int\limits_0^{y} \left[v(\lambda,\xi,t-y+\xi) + \int\limits_0^{t-y} k(\tau)v(\lambda,\xi,t-y+\xi-\tau)d\tau \right] d\xi +$$

$$+\frac{\lambda^2}{2}\int\limits_y^{(y+t)/2} \left[ak(t+y-2\xi) - v(\lambda,\xi,t+y-\xi) - \int\limits_0^{t+y-2\xi} k(\tau)v(\lambda,\xi,t+y-\xi-\tau)d\tau \right] d\xi, \quad (2.5)$$

$$v_y(\lambda,y,t) = -\frac{\lambda^2 a}{2}k(t-y)y + f(t-y) -$$

$$-\frac{\lambda^2}{2}\int\limits_0^{(t-y)/2} \left[ak(t-y-2\xi) - v(\lambda,\xi,t-y-\xi) - \int\limits_0^{t-y-2\xi} k(\tau)v(\lambda,\xi,t-y-\xi-\tau)d\tau \right] d\xi +$$

$$+\frac{\lambda^2}{2}\int\limits_0^{y} \left[v(\lambda,\xi,t-y+\xi) + \int\limits_0^{t-y} k(\tau)v(\lambda,\xi,t-y+\xi-\tau)d\tau \right] d\xi +$$

$$+\frac{\lambda^2}{2}\int\limits_y^{(y+t)/2} \left[ak(t+y-2\xi) - v(\lambda,\xi,t+y-\xi) - \int\limits_0^{t+y-2\xi} k(\tau)v(\lambda,\xi,t+y-\xi-\tau)d\tau \right] d\xi, \quad (2.6)$$

$$v(\lambda,y,t) = \frac{\lambda^2 a}{2} y \int\limits_y^t k(\tau-y)d\tau + \frac{\lambda^2 a}{2} \int\limits_y^t \int\limits_y^{(\tau+y)/2} k(\tau+y-2\xi)d\xi d\tau - \frac{1}{2}\int\limits_y^t f(\tau-y)d\tau + \frac{1}{2}v(\lambda,0,t-y) -$$

$$-\frac{\lambda^2}{2}\int\limits_y^t \int\limits_0^y \left[v(\lambda,\xi,\tau-y+\xi) + \int\limits_0^{\tau-y} k(\alpha)v(\lambda,\xi,\tau-y+\xi-\alpha)d\alpha \right] d\xi d\tau -$$

$$-\frac{\lambda^2}{2}\int\limits_y^t \int\limits_y^{(\tau+y)/2} \left[v(\lambda,\xi,\tau+y-\xi) + \int\limits_0^{\tau+y-2\xi} k(\alpha)v(\lambda,\xi,\tau+y-\xi-\alpha)d\alpha \right] d\xi d\tau. \quad (2.7)$$

The equation (2.7) is an integral equation of the Volterra type in the domain D_T and defines the unique continuous solution using the equations (2.5), (2.6) we conclud that this solution has a continuous derivative in D_T. Replacing the expression for $v(\lambda,0,t)$, determined by the formula (2.3), in the equation (2.7) and for equations (2.5)–(2.7) constructing the method of consecutive approximations on a usual scheme, that has a factorial convergence at the argument t, it is not difficult to deduce the estimate (2.1) in the domain D_T.

By means of the equation (2.5) we obtain the function

$$\psi(\lambda_1,\lambda_2,t)=\frac{\left(\lambda_1^2-\lambda_2^2\right)a}{2}+$$

$$+\int\limits_0^{t/2}\Big\{a(\lambda_1^2-\lambda_2^2)k(t-2\xi)-\lambda_1^2 v(\lambda_1,\xi,t-\xi)+\lambda_2^2 v(\lambda_2,\xi,t-\xi)-$$

$$-\int\limits_0^{t-2\xi}k(\tau)\Big[\lambda_1^2 v(\lambda_1,\xi,t-\xi-\tau)-\lambda_2^2 v(\lambda_2,\xi,t-\xi-\tau)\Big]d\tau\Big\}d\xi. \tag{2.8}$$

The right side of this equation belongs to the class $C^1[0,T]$. For this reason $\psi(\lambda_1,\lambda_2,t)\in C^1[0,T]$, at any fixed $\lambda_j, j=1,2$.

As a consequence of the Lemma we should note that under the conditions of the Lemma 2.1 the function $F_0(\lambda,t)$, in the equation (1.12), at each fixed λ belongs to the class $C^1[0,T]$, and the function $F_{00}(t)\equiv F_0(\lambda_1,t)-F_0(\lambda_2,t)$ belongs to the class $C^2[0,T]$. Besides, from the formulae (2.5), (2.8) it is concluded that the following equations are valid

$$F_0(\lambda,0)=0,\quad F_{0t}(\lambda,0)=\frac{\lambda^2 a}{2}-f(0).$$

From now on we will assume that the function $F_0(\lambda,t)$ satisfies these necessary conditions. □

3. Construction of a System Integral Equations for Equivalent Inverse Problems (1.8)–(1.10), (1.12)

Lemma 3.1. *Let the conditions of the lemma 2.1 and the necessary conditions relative to the function F_0 be met. Then the inverse problem (1.8)–(1.10), (1.12) in the domain D_T is equivalent to the problem of finding the functions v, v_t, k, f from the following system of integral equations:*

$$v(\lambda_j,y,t)=F_0(\lambda_j,t-y)+\frac{\lambda_j^2 a}{2}y+$$

$$+\frac{\lambda_j^2}{2}\int\limits_{D(y,t)}\left[ak(\tau-\xi)-v(\lambda_j,\xi,\tau)-\int\limits_0^{\tau-\xi}k(\alpha)v(\lambda_j,\xi,\tau-\alpha)d\alpha\right]d\tau d\xi,\ j=1,2, \tag{3.1}$$

$$v_t(\lambda_j,y,t)=F_{0t}(\lambda_j,t-y)+\frac{\lambda_j^2}{2}\int_0^{(t+y)/2}\Big\{ak(t-|y-\xi|-\xi)-v(\lambda_j,\xi,t-|y-\xi|)-$$

$$-\int\limits_0^{t-|y-\xi|-\xi}k(\alpha)v(\lambda_j,\xi,t-|y-\xi|-\alpha)d\alpha-\frac{1}{2}\Big[ak\left(\frac{t-y}{2}+\left|\xi-\frac{t-y}{2}\right|-\xi\right)-$$

$$-v\left(\lambda_j,\xi,\frac{t-y}{2}+\left|\xi-\frac{t-y}{2}\right|\right)+\int\limits_0^{\frac{t-y}{2}+|\xi-\frac{t-y}{2}|-\xi}k(\alpha)v\left(\lambda_j,\xi,\frac{t-y}{2}+\left|\xi-\frac{t-y}{2}\right|-\alpha\right)d\alpha\Bigg]\times$$

$$\times\left(1-\operatorname{sign}\left(\xi-\frac{t-y}{2}\right)\right)\Bigg\}d\xi, j=1,2, \tag{3.2}$$

$$k(t)=\frac{2}{(\lambda_1^2-\lambda_2^2)a}F_{00tt}(t)+\frac{\lambda_1^2+\lambda_2^2}{2}t+$$

$$+\frac{2}{(\lambda_1^2-\lambda_2^2)a}\int\limits_0^t\left[\frac{(\lambda_1^4-\lambda_2^4)a}{4}k(\xi)(t-\xi)+\lambda_1^2v_t\left(\lambda_1,\frac{t-\xi}{2},\frac{t+\xi}{2}\right)-\lambda_2^2v_t\left(\lambda_2,\frac{t-\xi}{2},\frac{t+\xi}{2}\right)+\right.$$

$$\left.+\int\limits_0^\xi k(\tau)\left[\lambda_1^2v_t\left(\lambda_1,\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)-\lambda_2^2v_t\left(\lambda_2,\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)\right]d\tau\right]d\xi. \tag{3.3}$$

$$f(t)=-F_{0t}(\lambda,t)+\frac{\lambda^2a}{2}+$$

$$+\frac{\lambda^2}{2}\int\limits_0^t\left[ak(\xi)-v\left(\lambda,\frac{t-\xi}{2},\frac{t+\xi}{2}\right)-\int\limits_0^\xi k(\tau)v\left(\lambda,\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)d\tau\right]d\xi, \tag{3.4}$$

where

$$D(y,t)=\{(\xi,\tau)|(t-y)/2+|\xi-(t-y)/2|<\tau<t-|\xi-y|, 0<\xi<(t+y)/2\},$$

$$F_{00}(t)\equiv F_0(\lambda_1,t)-F_0(\lambda_2,t).$$

Proof. First of all we observe that the system of integral equations (3.1)-(3.3) is closed in the domain D_T and uniquely determines continuous functions v, v_t, k. After finding the unknown function $f(t)$ is calculated by the formula (3.4). In the formula (3.4) for $\lambda=\lambda_1$ and $\lambda=\lambda_2$, the result of the calculation should not depend on the choice of the parameter λ. Further for definiteness in (3.4) we put $\lambda=\lambda_1$. we first consider the auxiliary problem of identifying the functions $\varphi(y,t)$, satisfying the relations [5]

$$\varphi_{tt}-\varphi_{yy}=\gamma(y,t),\quad 0<y<t<T-y,$$

$$\varphi(0,t)=\varphi_0(t),\quad 0\leqslant y\leqslant T,\quad \varphi(y,y)=\varphi_1(y),\quad 0\leqslant y\leqslant T,$$

where $\gamma,\varphi_0,\varphi_1$ are continuous functions and $\varphi_0(0)=\varphi_1(0)$. It is easy to check that the generalized solution of this problem is given by the formula

$$\varphi(y,t)=\varphi_0(t-y)+\varphi_1\left(\frac{t+y}{2}\right)-\varphi_1\left(\frac{t-y}{2}\right)+\frac{1}{2}\int\limits_{D(y,t)}\gamma(\xi,\tau)d\tau d\xi, \tag{3.5}$$

According to the formula (3.5) the solution of the problems (1.8), (1.9), (1.12) satisfies the integral equation (3.1) in the domain D_T. In order to obtain the equation (3.2) we reduce the integral over $D(y,t)$ to a repeated integral:

$$\int\limits_{D(y,t)}\gamma(\xi,\tau)d\tau d\xi=\int\limits_0^{(t+y)/2}\int\limits_{(t-y)/2+|\xi-(t-y)/2|}^{t-|y-\xi|}\gamma(\xi,\tau)d\tau d\xi.$$

Differentiating the equation (3.1) with respect to the variable t, we arrive to the equation (3.2). Differentiating the equation (3.1) by the variable y, we find

$$v_y(\lambda_j, y, t) = -F_{0t}(\lambda_j, t-y) + \frac{a}{2}\lambda_j^2 + \frac{1}{2}\lambda_j^2 \int\limits_0^{(t+y)/2} \Bigg\{ \Bigg[ak(t-|y-\xi|-\xi) - v(\lambda_j, \xi, t-|y-\xi|) -$$

$$-\int\limits_0^{t-|y-\xi|-\xi} k(\alpha) v(\lambda_j, \xi, t-|y-\xi|-\alpha) d\alpha \Bigg] \operatorname{sign}(\xi - y) + \frac{1}{2}\Bigg[ak(\frac{t-y}{2} + \left|\xi - \frac{t-y}{2}\right| - \xi) -$$

$$-v\left(\lambda_j, \xi, \frac{t-y}{2} + \left|\xi - \frac{t-y}{2}\right|\right) - \int\limits_0^{\frac{t-y}{2}+|\xi-\frac{t-y}{2}|-\xi} k(\alpha) v(\lambda_j, \xi, \frac{t-y}{2} +$$

$$+\left|\xi - \frac{t-y}{2}\right| - \alpha) d\alpha \Bigg] \left(1 - \operatorname{sign}\left(\xi - \frac{t-y}{2}\right)\right) \Bigg\} d\xi, j = 1, 2. \tag{3.6}$$

Assuming in this equation that $y = 0$ and using the equation (1.10) we obtain the equation (3.4). Considering the equation (1.10) for $\lambda = \lambda_j,\ j = 1, 2$ and constructing the difference, we obtain

$$v_y(\lambda_1, 0, t) - v_y(\lambda_2, 0, t) = 0. \tag{3.7}$$

From the equation (3.7), with the use of the equation (3.6) at $y = 0$, we find

$$F_{0t}(\lambda_1, t) - F_{0t}(\lambda_2, t) = \frac{(\lambda_1^2 - \lambda_2^2)a}{2} + \int\limits_0^{t/2} \Bigg\{ a(\lambda_1^2 - \lambda_2^2) k(t - 2\xi) - \lambda_1^2 v(\lambda_1, \xi, t-\xi) +$$

$$+\lambda_2^2 v(\lambda_2, \xi, t-\xi) - \int\limits_0^{t-2\xi} k(\tau) \Big[\lambda_1^2 v(\lambda_1, \xi, t-\xi-\tau) - \lambda_2^2 v(\lambda_2, \xi, t-\xi-\tau) \Big] d\tau \Bigg\} d\xi. \tag{3.8}$$

Differentiating, this equation with respect to the variable t, after simple calculations we obtain the equations (3.3). We note that the unknown function $f(t)$ does not belong to the equation (3.1)–(3.3).

It is not difficult to check that the inverse transform exists, too. Indeed, integrating the equation (3.2) by t within the interval from y to t and changing, where necessary, the order of integration we obtain the equation (3.1), from which, in turn, the relations (1.8), (1.9), (1.12) for $\lambda = \lambda_j$ are concluded. In the integrals on the right side (3.3) we make a substitution of the variable ξ by ξ' according to the formula $\xi' = (t-\xi)/2$. Then at the right and left sides of this equation substituting t by $t - 2\tau$ we multiply both sides by $d\tau$ and integrate according to τ in the interval of 0 to $t/2$. In the repeated integrals of the formed equation we change the order of the integration. After simple calculations we arrive at the equation (3.8). The equivalence of the equations (3.4) and (1.10) is proved the same way. □

4. Formulation and Proof of the Main Results

We denote by $A(k_0)$ the set of pairs of functions $\{k(t), f(t)\}$, that for some $T > 0$ satisfy the following condition

$$\max\{\|k\|_{C[0,T]}, \|f\|_{C[0,T]}\} \leqslant k_0,\ k_0 > 0.$$

Theorem 4.1. *Let $(k^{(1)}, f^{(1)}) \in A(k_0)$, $(k^{(2)}, f^{(2)}) \in A(k_0)$ of the solution of the inverse problem (1.8)–(1.10),(1.12) with the data $F_0^{(1)}(\lambda_j, t)$, $F_0^{(2)}(\lambda_j, t)$, $j = 1, 2$, respectively, $F_{00}^{(i)} \equiv F_0^{(i)}(\lambda_1, t) - F_0^{(i)}(\lambda_2, t)$. Then there exists a positive constant C, depending on $\lambda_1, \lambda_2, T, k_0, a$, such that the following estimate is valid*

$$\|k^{(1)} - k^{(2)}\|_{C[0,T]} + \|f^{(1)} - f^{(2)}\|_{C[0,T]} \leqslant$$

$$\leqslant C\left(\sum_{j=1}^{2} \|F_0^{(1)}(\lambda_j, t) - F_0^{(2)}(\lambda_j, t)\|_{C^1[0,T]} + \|\tilde{F}_{00}^{(1)} - \tilde{F}_{00}^{(2)}\|_{C^2[0,T]}\right). \tag{4.1}$$

Uniqueness the following theorem easily follows from the Theorem 4.1 for any $T > 0$.

Theorem 4.2. *Let the function $(k^{(i)}, f^{(i)}) \in C[0,T]$ and $F_0^{(i)}(\lambda_j, t)$, $i = 1, 2$, $j = 1, 2$ has the same meaning as in the theorem 4.1. If in this $F_0^{(1)}(\lambda_j, t) = F_0^{(2)}(\lambda_j, t)$, $j = 1, 2$ for $t \in [0, T]$, then $k^{(1)} = k^{(2)}$, $f^{(1)} = f^{(2)}$ at $t \in [0, T]$.*

We assume that the functions $f^{(i)}$, $F^{(i)}(\lambda_j, t)$, $k^{(i)}$, $i = 1, 2$, $j = 1, 2$ are the same as in the theorem 4.1. The solution of the problems (1.8)–(1.10) at $k = k^{(i)}$, $f = f^{(i)}$, $\lambda = \lambda_j$ will be denoted by $v^{(ij)}(y, t)$. We also introduce the following notation:

$$\tilde{f}(t) = f^{(1)} - f^{(2)},\ \ \tilde{F}_0(\lambda_j, t) = F_0^{(1)}(\lambda_j, t) - F_0^{(2)}(\lambda_j, t),\ \ \tilde{k}(t) = k^{(1)} - k^{(2)},$$

$$\tilde{v}^{(j)}(y, t) = v^{(1j)}(y, t) - v^{(2j)}(y, t),\ j = 1, 2.$$

We write for newly introduced function the integral relations corresponding to it. From the equations (3.1) and (3.2) it is concluded that

$$\tilde{v}^{(j)}(y, t) = \tilde{F}_0(\lambda_j, t - y) + \frac{\lambda_j}{2}\int_{D(y,t)} \Big\{ a\tilde{k}(\tau - \xi) - \tilde{v}^{(j)}(\xi, \tau) -$$

$$- \int_0^{\tau - \xi} \Big[\tilde{k}(\alpha) v^{(1j)}(\xi, \tau - \alpha) + k^{(2)}(\alpha)\tilde{v}^{(j)}(\xi, \tau - \alpha)\Big] d\alpha \Big\} d\tau d\xi, \tag{4.2}$$

$$\tilde{v}_t^{(j)}(y, t) = \tilde{F}_{0t}(\lambda_j, t - y) + \frac{\lambda_j}{2}\int_0^{(t+y)/2} \Big\{ a\tilde{k}(t - |y - \xi| - \xi) - \tilde{v}^j(\xi, t - |y - \xi|) -$$

$$- \int_0^{t-|y-\xi|-\xi} \Big[\tilde{k}(\alpha) v^{(1j)}(\xi, t - |y - \xi| - \alpha) + k^{(2)}(\alpha)\tilde{v}^{(j)}(\xi, t - |y - \xi| - \alpha)\Big] d\alpha -$$

$$- \frac{1}{2}\Big[a\tilde{k}\left(\frac{t-y}{2} + \left|\xi - \frac{t-y}{2}\right| - \xi\right) - \tilde{v}^j\left(\xi, \frac{t-y}{2} + \left|\xi - \frac{t-y}{2}\right|\right) -$$

$$- \int_0^{(\frac{t-y}{2} + |\xi - \frac{t-y}{2}| - \xi)} \left(\tilde{k}(\alpha) v^{(1j)}\left(\xi, \frac{t-y}{2} + \left|\xi - \frac{t-y}{2}\right| - \alpha\right) +\right.$$

$$+k^{(2)}(\alpha)\tilde{v}^{(j)}\left(\xi,\frac{t-y}{2}+\left|\xi-\frac{t-y}{2}\right|-\alpha\right)\right)d\alpha\Big]\left(1-\operatorname{sign}\left(\xi-\frac{t-y}{2}\right)\right)\Big\}d\xi. \qquad (4.3)$$

From the relations (3.3), (3.4) we find the equations for the functions $\tilde{f}(t)$, $\tilde{k}(t)$ in the form of

$$\tilde{k}(t)=\frac{2}{(\lambda_1^2-\lambda_2^2)a}\left[\tilde{F}^1_{0tt}(\lambda_1,t)-\tilde{F}^1_{0tt}(\lambda_2,t)\right]+$$
$$+\frac{2}{(\lambda_1^2-\lambda_2^2)a}\int_0^t\left\{\frac{(\lambda_1^4-\lambda_2^4)a}{4}\tilde{k}(\xi)(t-\xi)+\lambda_1^2\tilde{v}_t^{(1)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}\right)-\lambda_2^2\tilde{v}_t^{(2)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}\right)+\right.$$
$$+\int_0^\xi\left[\tilde{k}(\tau)\left(\lambda_1^2v_t^{(11)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)-\lambda_2^2v_t^{(12)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)\right)+\right.$$
$$\left.\left.+k^2(\tau)\left(\lambda_1^2\tilde{v}_t^{(1)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)-\lambda_2^2\tilde{v}_t^{(2)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)\right)\right]d\tau\right\}d\xi. \qquad (4.4)$$

$$\tilde{f}(t)=-\tilde{F}_{0t}(\lambda,t)+\frac{\lambda_1^2}{2}\int_0^t\left\{a\tilde{k}(\xi)-\tilde{v}^{(1)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}\right)-\right.$$
$$\left.-\int_0^\xi\left[\tilde{k}(\tau)v^{(11)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)+k^{(2)}(\tau)\tilde{v}^{(1)}\left(\frac{t-\xi}{2},\frac{t+\xi}{2}-\tau\right)\right]d\tau\right\}d\xi. \qquad (4.5)$$

We denote

$$\rho(t)=\max\left[\max_{0\leqslant y\leqslant T/2-|t-T/2|}|\tilde{v}^{(j)}(y,t)|,\max_{0\leqslant y\leqslant T/2-|t-T/2|}|\tilde{v}_t^{(j)}(y,t)|,|\tilde{k}(t)|,|\tilde{f}(t)|\right],$$
$$t\in[0,T],\quad j=1,2.$$

According to the Lemma 2.1 the functions v^{ij} are differentiable in the domain D_T and satisfy the estimate

$$\|v^{ij}\|_{C^1(D_T)}\leqslant M,\quad i,j=1,2,$$

with some constant M, depending only on $\lambda_1,\lambda_2,T,k_0,a$. Using an equivalent definition of the domain D_T in the form of

$$D_T=\Big\{(\xi,\tau)||\tau-t+y|\leqslant\xi\leqslant(t+y)/2-|(t+y)/2-\tau|,(t-y)/2\leqslant\tau\leqslant t\Big\},$$

from the equation (4.2) we find that for $y\in[0,T/2-|t-T/2|]$ the following inequalities are valid

$$|\tilde{v}^{(j)}(y,t)|\leqslant\|\tilde{F}_0(\lambda_j,t)\|_{C[0,T]}+\frac{\lambda_j^2}{2}\int\limits_{D(y,t)}\left\{a\rho(\tau-\xi)+\rho(\tau)+\int\limits_0^{\tau-\xi}(M\rho(\alpha)+k_0\rho(\tau-\alpha))d\alpha\right\}d\tau d\xi\leqslant$$
$$\leqslant\|\tilde{F}_0(\lambda_j,t)\|_{C[0,T]}+\frac{T\lambda_j^2}{4}\int\limits_0^t\left\{(1+a)\rho(\tau)+(M+k_0)\int\limits_0^\tau\rho(\alpha)d\alpha\right\}d\tau\leqslant$$
$$\leqslant\|\tilde{F}_0(\lambda_j,t)\|_{C[0,T]}+\frac{T\lambda_j^2}{4}((1+a)+T(M+k_0))\int\limits_0^t\rho(\tau)d\tau. \qquad (4.6)$$

Analogously, we obtain

$$|\tilde{v}_t^{(j)}(y,t)| \leqslant \|\tilde{F}_{0t}(\lambda_j,t)\|_{C[0,T]} + c_1 \int\limits_0^t \rho(\tau)d\tau, \tag{4.7}$$

$$|\tilde{k}(t)| \leqslant \frac{2}{(\lambda_1^2-\lambda_2^2)a}\|\tilde{F}_{00tt}^{(1)} - \tilde{F}_{00tt}^{(2)}\|_{C[0,T]} + c_2 \int\limits_0^t \rho(\tau)d\tau, \tag{4.8}$$

$$|\tilde{f}(t)| \leqslant \|\tilde{F}_{0t}(\lambda_1,t)\|_{C[0,T]} + c_3 \int\limits_0^t \rho(\tau)d\tau, \tag{4.9}$$

where the constants $c_i, i = 1,2,3$ only depend on $\lambda_1,\lambda_2,T,k_0,a$. From the relations (4.6)–(4.9) it is concluded that $\rho(t)$ satisfies an integral inequality

$$\rho(t) \leqslant \max\Big\{ \|\tilde{F}_0(\lambda_j,t)\|_{C[0,T]}, \|\tilde{F}_{0t}(\lambda_j,t)\|_{C[0,T]},$$

$$\frac{2}{(\lambda_1^2-\lambda_2^2)a}\|\tilde{F}_{00tt}^{(1)} - \tilde{F}_{00tt}^{(2)}\|_{C[0,T]}, \|\tilde{F}_{0t}(\lambda_1,t)\|_{C[0,T]}\Big\} + c_0 \int\limits_0^t \rho(\tau)d\tau,$$

with some constant c_0, that only depends on $\lambda_1,\lambda_2,T,k_0,a$.

It is well-known that the last inequality admits the estimate (4.1).

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