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On Interpolation in the Class of Analytic Functions in the Unit Disk with Power Growth of the Nevanlinna Characteristic

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In this paper we solve the interpolation problem for the class of analytic functions in the unit disk with power growth of the Nevanlinna characteristic under the condition that interpolation nodes are contained in a finite union of Stolz angles.

Keywords: interpolation, holomorphic functions, the Nevanlinna characteristic, Stolz angles.

Introduction

Let $\mathbb{C}$ be the complex plane, D be the unit disk on $\mathbb{C}$, $H(D)$ be the set of all functions, holomorphic in D. For any $\alpha > 0$ we define the class S_α^∞ as:

$$S_\alpha^\infty := \left\{ f \in H(D) : T(r, f) \leqslant \frac{C_f}{(1-r)^\alpha} \right\},$$

where $C_f > 0$ is a positive constant, depending on the function f, $r \in [0, 1)$, $T(r, f) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \ln^+ |f(re^{i\varphi})| d\varphi$ is the Nevanlinna characteristic of the function f, $\ln^+ |a| = \max(0, \ln |a|)$, $a \in \mathbb{C}$ (see [1]).

It is well known that if $f \in S_\alpha^\infty$, then

$$M(r, f) = \max_{|z| \leqslant r} |f(z)| \leqslant \exp \left\{ \frac{c_f}{(1-r)^{\alpha+1}} \right\} \tag{1}$$

for all $\alpha > 0$, $c_f > 0$ (see [1]).

It is clear that if $f \in S_\alpha^\infty$ and $\{\alpha_k\}_{k=1}^{+\infty}$ is a sequence of points from the unit disk, then the operator $R(f) = (f(\alpha_1), ..., f(\alpha_k), ...)$ maps the class S_α^∞ into the class of weighted sequences

$$l_\alpha = \left\{ \gamma = \{\gamma_k\}_{k=1}^{+\infty} : |\gamma_k| \leqslant \exp \frac{\lambda}{(1 - |\alpha_k|)^{\alpha+1}},\ \lambda > 0 \right\}.$$

In this article we answer the question under what conditions on the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ the operator $R(f)$ maps the class S_α^∞ *onto* the class l_α.

Definition 1. *A sequence $\{\alpha_k\}_{k=1}^{+\infty}$ is called interpolating for S_α^∞, if $R(S_\alpha^\infty) = l_\alpha$.*

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Let us note that interpolation theory has become intensively developed since the fundamental work of L. Carleson (see [2]) about interpolation in the class of bounded analytic functions. The term of "free interpolation" was first introduced in [3]. The interpolation problem in subclasses of the bounded type functions N was investigated there. This problem in the Nevanlinna and Smirnov classes was solved in [4, 5]. These questions in the Hardy and Bergman spaces was studied in works [6, 7].

The paper is organized as follows: in the first section we present the formulation of main result of the article and prove some auxiliary results, in the second section we present the proof of main result.

1. Formulation of main result and proof of auxiliary results

To formulate and proof the results of the work we introduce some more notation and definitions:

For any $\beta > -1$ we denote $\pi_\beta(z, \alpha_k)$ as M. M. Djrbashian's infinite product with zeros at points of the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ (see [8]):

$$\pi_\beta(z, \alpha_k) = \prod_{k=1}^{+\infty} \left(1 - \frac{z}{\alpha_k}\right) \exp(-U_\beta(z, \alpha_k)), \tag{2}$$

where

$$U_\beta(z, \alpha_k) = \frac{2(\beta+1)}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2)^\beta \ln|1 - \frac{\rho e^{i\theta}}{\alpha_k}|}{(1 - z\rho e^{-i\theta})^{\beta+2}} \rho\, d\rho\, d\theta. \tag{3}$$

We denote $\pi_{\alpha,n}(z, \alpha_k)$ as infinite product $\pi_\beta(z, \alpha_k)$ without n-th factor.

As stated in [8], the infinite product $\pi_\beta(z, \alpha_k)$ is absolutely and uniformly convergent in the unit disk D if and only if the series converges:

$$\sum_{k=1}^{+\infty} (1 - |\alpha_k|)^{\beta+2} < +\infty. \tag{4}$$

Let us remark that the product $\pi_\beta(z, \alpha_k)$ appear naturally in the integral representations of the holomorphic functions by the kernel

$$K_\alpha(\zeta, z) = \frac{\alpha+1}{\pi} \frac{(1-|\zeta|^2)^\alpha}{(1-\bar{\zeta}z)^{\alpha+2}}, \zeta, z \in D,$$

as there is the Blaschke product in the integral representation of the bounded type functions by the Poisson-Jensen formula (see [1, 9]).

If $\beta = p \in \mathbb{Z}_+$ then product (2) takes a form (see [8]):

$$\pi_p(z, \alpha_k) = \prod_{k=1}^{+\infty} \frac{\overline{\alpha}_k(\alpha_k - z)}{1 - \overline{\alpha}_k z} \exp \sum_{j=1}^{p+1} \frac{1}{j} \left(\frac{1 - |\alpha_k|^2}{1 - \overline{\alpha}_k z}\right)^j.$$

Definition 2. *The part of the angle with vertex at the point $e^{i\theta}$, less then π and contained in D, whose bisector coincides with radius connecting the center of the disk and the point $e^{i\theta}$ is said to be the Stolz angle with vertex at the point $e^{i\theta}$, i.e.*

$$\Gamma_\delta(\theta) := \{z \in D : |\arg(e^{i\theta} - z) - \theta| \leqslant \frac{\pi\delta}{2}\},$$

where $0 \leqslant \delta < 1$.

The main result of this article is the proof of the following theorem:

Theorem 1.1. *Let $\{\alpha_k\}_{k=1}^{+\infty}$ be the arbitrary sequence of complex numbers from D, which is contained in a finite union of Stolz angles, i.e.*

$$\{\alpha_k\} \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s), \tag{5}$$

with certain $0 < \delta < \dfrac{1}{\alpha+1}$.
The following statements are equivalent:
i) $\{\alpha_k\}_{k=1}^{+\infty}$ *is an interpolating sequence in* S_α^∞, $\alpha > 0$,
ii)

$$n(r) = \{card\, \alpha_k : |\alpha_k| < r < 1\} \leqslant \frac{c}{(1-r)^{\alpha+1}}, \tag{6}$$

for some $c > 0$*;*

$$|\pi'_\beta(\alpha_n, \alpha_k)| \geqslant \exp \frac{-M}{(1-|\alpha_n|)^{\alpha+1}}, \tag{7}$$

for some $M > 0$ *and all* $\beta > \alpha - 1$.

For presentation of auxiliary result we need also the O. Besov class $B_{1,\infty}^s$ on the unite circle (see [10, p. 151]). Let $0 < s < 2$; function ψ integrable on the unite circle belongs to the class $B_{1,\infty}^s$ if and only if

$$\sup_{0<t<1} \left\{ \int_{-\pi}^{\pi} \frac{|\psi(e^{i(\theta+t)}) - 2\psi(e^{i\theta}) + \psi(e^{i(\theta-t)})|}{|t|^s} d\theta \right\} < +\infty.$$

The proof of the theorem is based on the following statements.

Theorem A.(see [11]). *The class S_α^∞ coincides with the class of analytic in D functions represented as*

$$f(z) = c_\lambda z^\lambda \pi_\beta(z, \alpha_k) \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\psi(e^{i\theta})}{(1-ze^{-i\theta})^{\beta+2}} d\theta \right\}, \ z \in D, \tag{8}$$

for all $\beta > \alpha - 1$, where $\psi(e^{i\theta})$ is real-valued function from the O. Besov class $B_{1,\infty}^{\beta-\alpha+1}$, $\lambda \in \mathbb{Z}_+$, $c_\lambda \in \mathbb{C}$, and the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ satisfies the condition

$$n(r, f) \leqslant \frac{c_f}{(1-r)^{\alpha+1}}$$

for some $c_f > 0$.

Here and in the sequel, unless otherwise noted, we denote by c, $c_1, ..., c_n(\alpha, \beta, ...)$ some arbitrary positive constants depending on α, $\beta, ...$, whose specific values are immaterial.

It is clear that the space l_α coincides with the space of sequences $\{\gamma_k\}_{k=1}^{+\infty}$ such that

$$\sup_{k\geqslant 1} \left\{ (1-|\alpha_k|)^{\alpha+1} \ln(1+|\gamma_k|) \right\} < +\infty,$$

where $\{\alpha_k\}_{k=1}^{+\infty} \subset D$.

For the further exposition of the results we introduce metrics in spaces S_α^∞ and l_α as follows: $\forall f, g \in S_\alpha^\infty$

$$\rho_{S_\alpha^\infty}(f, g) = \sup_{0<r<1} \left\{ (1-r)^\alpha \int_{-\pi}^{\pi} \ln(1 + |f(re^{i\theta}) - g(re^{i\theta})|) d\theta \right\},$$

$\forall a = \{a_k\}, b = \{b_k\} \in l_\alpha$

$$\rho_{l_\alpha}(a,b) = \sup_{k\geqslant 1}\left\{(1-|\alpha_k|)^{\alpha+1}\ln(1+|a_k-b_k|)\right\}.$$

It is easy to check that S_α^∞ and l_α are complete metric spaces with respect to these metrics, and the space S_α^∞ is invariant regarding the shift.

Lemma 1.2. *If the operator $R(f) = (f(\alpha_1), ..., f(\alpha_n), ...)$ maps space S_α^∞ onto space l_α, then there exists the sequence of the functions $\{g_n(z)\}_{n=1}^{+\infty} \in S_\alpha^\infty$ such that*

$$\sup_{n\geqslant 1}\rho_{S_\alpha^\infty}(g_n, 0) \leqslant C, \quad C > 0$$

and

$$g_n(\alpha_k) = \gamma_k^{(n)}, \text{ where } \gamma_k^{(n)} = \begin{cases} 0, \text{ for all } k \neq n, \\ \exp\dfrac{1}{(1-|\alpha_k|)^{\alpha+1}}, \text{ for } k = n, \end{cases}$$

where $k, n = 1, 2, ...$

Proof. Let S_L, $L > 0$, be the set of the sequences $c = \{c_k\}_{k=1}^{+\infty}$ from the space l_α such that $c_k = F(\alpha_k)$, $k = 1, 2, ...$ for certain function $F \in S_\alpha^\infty$, satisfying condition $\rho_{S_\alpha^\infty}(F, 0) \leqslant L$, that is

$$\sup_{0<r<1}\left\{(1-r)^\alpha \int_{-\pi}^{\pi}\ln(1+|F(re^{i\theta})|)d\theta\right\} \leqslant L.$$

By hypothesis, $R(S_\alpha^\infty) = l_\alpha$, equivalently to $l_\alpha = \cup_{L=1}^{+\infty} S_L$. Let prove that the sets S_L are closed for all L in l_α, in this terms if $c^{(m)} = \{c_k^{(m)}\}_{k=1}^{+\infty} \in S_L$ and $\rho_{l_\alpha}(c^{(m)}, c^{(0)}) \to 0$ as $m \to +\infty$, then $c^{(0)} \in S_L$.

By assumption, $c_k^{(m)} = F_m(\alpha_k),\ k = 1, 2, ...,\ F_m \in S_\alpha^\infty$ with

$$\rho_{S_\alpha^\infty}(F_m, 0) \leqslant L \tag{9}$$

and $\rho_{l_\alpha}(c^{(m)}, c^{(0)}) \to 0$ as $m \to +\infty$.

In this notation, we need to prove that there exists function $F_0 \in S_\alpha^\infty$, such that $F_0(\alpha_k) = c_k^{(0)}$, $k = 1, 2, ...$ and $\rho_{S_\alpha^\infty}(F_0, 0) \leqslant L$.

From (9) it follows that for any $m = 1, 2, ...$

$$\ln^+|F_m(re^{i\theta})| \leqslant \frac{C(L)}{(1-r)^{\alpha+1}},$$

where $C(L)$ is independent on m.

By Montel's theorem we can choose the subsequence of functions $\{F_{m_k}(z)\}$, uniformly convergent to function F_0 on compact subsets of the unit disk. Let check that $F_0 \in S_\alpha^\infty$.

Using inequality (9), we get:

$$\sup_{0<r\leqslant R<1}\left\{(1-r)^\alpha \int_{-\pi}^{\pi}\ln(1+|F_{m_k}(re^{i\theta})|)d\theta\right\} \leqslant L.$$

Taking in this inequality the limit as $k \to +\infty$, we obtain:

$$\sup_{0<r\leqslant R<1}\left\{(1-r)^\alpha \int_{-\pi}^{\pi}\ln(1+|F_0(re^{i\theta})|)d\theta\right\} \leqslant L.$$

Whence guiding $R \to 1-0$, we have:

$$\rho_{S_\alpha^\infty}(F_0, 0) \leqslant L. \tag{10}$$

Thus, $F_0 \in S_\alpha^\infty$.

Since $F_{m_k}(z) \to F_0(z)$ as $k \to +\infty$ for all $z \in D$, we have $c_n^{(m_k)} = F_{m_k}(\alpha_n) \to F_0(\alpha_n) = c_n^{(0)}$ as $k \to +\infty$ for all $|\alpha_n| < 1$, $n = 1, 2, \ldots$. Taking into account the estimate (10), we conclude: $c^{(0)} = \{c_n^{(0)}\}_{n=1}^{+\infty} \in S_L$. Thus, we prove that the set S_L is closed in l_α and $l_\alpha = \cup_{L=1}^{+\infty} S_L$.

By Baire's theorem there exists a number L_0 such that the set S_{L_0} includes the ball with the center at one of its interior points, for example,

$$B(F_0(\alpha_n), d) := \{c = \{c_k\} : \rho_{l_\alpha}(c, F_0(\alpha_n)) = \sup_{k \geqslant 1}(1 - |\alpha_k|)^{\alpha+1} \ln(1 + |c_k - F_0(\alpha_k)|) \leqslant d\},$$

where the sequence $\{F_0(\alpha_n)\} = \{c_n^{(0)}\}$ is the center of the ball, $d > 0$ is its radius.

It means that for any sequence $c = \{c_k\} \in B$ there exists a function $F \in S_\alpha^\infty$ such that $F(\alpha_k) = c_k$, $k = 1, 2, \ldots$.

So, if $\rho_{l_\alpha}(\gamma, 0) \leqslant d$ for some $\gamma = \{\gamma_k\}$, then there exists a function $g \in S_\alpha^\infty$ such that $\rho_{S_\alpha^\infty}(g, 0) \leqslant 2L_0$ and $g(\alpha_k) = \gamma_k$ for all k. It is sufficient to put $\gamma_k = c_k - F_0(\alpha_k)$, $k = 1, 2, \ldots$, $g(z) = F(z) - F_0(z)$, where F is the function with the above properties.

Let $\gamma^{(n)} = \{\gamma_k^{(n)}\}$, $n \in \mathbb{N}$, where $\gamma_k^{(n)} = c_k^{(n)} - F_0(\alpha_k)$ $k = 1, 2, \ldots$, $c_k^{(n)} \in B$. Then $\rho_{l_\alpha}(\gamma^{(n)}, 0) \leqslant d$. We put $c_k^{(n)} = F_0(\alpha_k)$ for $k \neq n$, $c_k^{(n)} = F_0(\alpha_k) + \exp\dfrac{1}{(1 - |\alpha_k|)^{\alpha+1}}$ for $k = n$. Arguing as above, we see that there exists a function $g_n(z) = F_n(z) - F_0(z)$ such that $\rho_{S_\alpha^\infty}(g_n, 0) \leqslant 2L_0$ and $g_n(\alpha_k) = \gamma_k^{(n)} = 0$ for $k \neq n$, $g_n(\alpha_k) = \gamma_k^{(n)} = \exp\dfrac{1}{(1 - |\alpha_k|)^{\alpha+1}}$ for $k = n$. The proof of Lemma 1.2 is complete with the constant $C = 2L_0$.

Remark 1.1. The idea of proving Lemma 1.2 is adopted from P. Koosis (see [12, p. 200]), who first applied this for solving the interpolation problem in the class of the bounded analytic functions.

Lemma 1.3. *(see [13]) If members of the sequence $\{\alpha_k\}_{k=1}^{+\infty}$ are contained in a finite union of Stolz angles, i.e. $\{\alpha_k\} \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s)$ for certain $0 < \delta < \dfrac{1}{\alpha+1}$, then for any function $g(z) = \prod_{s=1}^{n} \exp\dfrac{C}{(1 - ze^{-i\theta_s})^{\alpha+1}}$, $z \in D$, $\alpha > -1$ the following estimate is valid*

$$|g(\alpha_s)| \geqslant c_0 \exp\frac{C}{(1 - |\alpha_s|)^{\alpha+1}}, \quad s = 1, 2, \ldots, n, \tag{11}$$

where c_0, C are some positive constants.

2. Proof of main result

Let prove the implication i) $\to$ ii).

We assume that $\{\alpha_k\}_{k=1}^{+\infty} \in D$ is an interpolation sequence in the class S_α^∞, $\alpha > 0$, i.e. for any $\{\gamma_k\} \in l_\alpha$ there exists a function $f \in S_\alpha^\infty$ such that $f(\alpha_k) = \gamma_k$, $k = 1, 2, \ldots$.

Let consider the sequence $\{\gamma_k\}_{k=1}^{+\infty}$: $\gamma_1 = 1, \gamma_2 = \gamma_3 = \ldots = 0$. Evidently, $\{\gamma_k\}_{k=1}^{+\infty} \in l_\alpha$. Since $\{\alpha_k\}_{k=2}^{+\infty}$ is zero-sequence for the function $f \in S_\alpha^\infty$, $\alpha > 0$, we have

$$n(r) \leqslant \frac{c}{(1 - r)^{\alpha+1}}$$

by Theorem A. The estimate (6) is established.

In order to show (7) we fix $n \in \mathbb{N}$ and take the sequence $\{\gamma_k^{(n)}\}_{k=1}^{+\infty}$ as follows: $\gamma_k^{(n)} = 0$, $k \neq n$, $\gamma_k^{(n)} = \exp \dfrac{1}{(1-|\alpha_k|)^{\alpha+1}}$, $k = n$. By Lemma 1.2 there exists a function $g_n \in S_\alpha^\infty$ such that $\rho_{S_\alpha^\infty}(g_n, 0) \leqslant C$ and $g_n(\alpha_k) = \gamma_k^{(n)}$ for all $k = 1, 2, ...$, where the constant $C > 0$ is independent on n. In particular, $g_n(\alpha_n) = \gamma_n^{(n)}$. According to Theorem A, any function $g_n \in S_\alpha^\infty$, $\alpha > 0$ can be represented as

$$g_n(z) = c_{\lambda_n} z^{\lambda_n} \pi_{\beta,n}(z, \alpha_k) \exp\{h_n(z)\}, \; z \in D,$$

where $h_n(z) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \frac{\psi_n(e^{i\theta})}{(1 - ze^{-i\theta})^{\beta+2}} d\theta$, $\beta > \alpha - 1$.

So,

$$|g_n(\alpha_n)| = |\gamma_n^{(n)}| = \exp \frac{1}{(1-|\alpha_n|)^{\alpha+1}} = |c_\lambda||\alpha_n|^\lambda |\pi_{\beta,n}(\alpha_n, \alpha_k)||\exp\{h(\alpha_n)\}|.$$

Since $\exp\{h_n(z)\} \in S_\alpha^\infty$, then taking into account the estimate (1) we have:

$$|g_n(\alpha_n)| = \exp \frac{1}{(1-|\alpha_n|)^{\alpha+1}} \leqslant c_1 |\pi_{\beta,n}(\alpha_n, \alpha_k)| \exp \frac{c_2}{(1-|\alpha_n|)^{\alpha+1}},$$

where c_2 is independent on n according to Lemma 1.2.

From the last inequality we obtain:

$$|\pi_{\beta,n}(\alpha_n, \alpha_k)| \geqslant \exp \frac{-M_1}{(1-|\alpha_n|)^{\alpha+1}}, \; M_1 > 0. \tag{12}$$

We show, that (12)⇒(7). Really, differentiating the function π_β, we get:

$$\pi'_\beta(z, \alpha_k) = \sum_{j=1}^{+\infty} \left(-\frac{1}{\alpha_j} \exp(-U_\beta(z, \alpha_j)) - \left(1 - \frac{z}{\alpha_j}\right) \exp(-U_\beta(z, \alpha_j)) U'_\beta(z, \alpha_j)) \right) \times \pi_{\beta,j}(z, \alpha_k),$$

where $U_\beta(z, \alpha_j)$ defined by (3).

Since $\pi_{\beta,j}(\alpha_n, \alpha_k) = \displaystyle\prod_{k=1, k\neq j}^{+\infty} \left(1 - \frac{\alpha_n}{\alpha_k}\right) \exp(-U_\beta(\alpha_n, \alpha_k)) = 0$ for all $j = 1, 2, ..., j \neq n$, we have:

$$|\pi'_\beta(\alpha_n, \alpha_k)| = \frac{1}{|\alpha_n|} |\exp(-U_\beta(\alpha_n, \alpha_n))| \, |\pi_{\beta,n}(\alpha_n, \alpha_k)|. \tag{13}$$

Using now estimate (12), from (3) and (13) we obtain:

$$|\pi'_\beta(\alpha_n, \alpha_k)| \geqslant \exp \frac{-M}{(1-|\alpha_n|)^{\alpha+1}}, \; M > 0.$$

Thus, (12)⇒(7). The implication i) → ii) is established.

Now we prove that ii) → i). Suppose that $\{\alpha_k\}_{k=1}^{+\infty}$ be the arbitrary sequence of complex numbers from D, which is contained in a finite union of Stolz angles, and the estimates (6), (7) are valid. Let us show that there exists a function $\Psi \in S_\alpha^\infty$ such that $\Psi(\alpha_k) = \gamma_k$, $k = 1, 2, ...$ for each $\{\gamma_k\}_{k=1}^{+\infty} \in l_\alpha$.

We construct a function $\Psi(z)$ as follows:

$$\Psi(z) = \sum_{k=1}^{+\infty} \gamma_k \frac{\pi_\beta(z, \alpha_j)}{(z - \alpha_k)} \frac{1}{\pi'_\beta(\alpha_k, \alpha_j)} \left(\frac{1 - |\alpha_k|}{1 - \overline{\alpha_k} z} \right)^m \frac{f(z)}{f(\alpha_k)}, \tag{14}$$

with $\beta > \alpha - 1$, $m > \alpha + 1$, and

$$f(z) = \prod_{s=1}^{n} \exp \frac{C}{(1 - ze^{-i\theta_s})^{\alpha+1}}, \ z \in D. \tag{15}$$

It is obvious, that $\Psi(\alpha_n) = \gamma_n$, $n = 1, 2, ...$

Now we need to prove that function $\Psi(z)$ is analytic in D and $\Psi \in S_\alpha^\infty$.

First, from the estimate (6) we conclude:

$$\sum_{k=1}^{+\infty} (1 - |\alpha_k|)^m < +\infty \tag{16}$$

for all $m > \alpha + 1$. Taking into account the convergence of the series (16) and Lemma 1.3, we obtain that the infinite product $\pi_\beta(z, \alpha_j)$ and the series (14) are absolutely and uniformly convergent in D.

Now we get an upper estimate for the function $|\Psi(z)|$. Since $\{\gamma_k\}_{k=1}^{+\infty} \in l_\alpha$ and condition (7) is valid, we have:

$$|\Psi(z)| \leqslant \sum_{k=1}^{+\infty} |\gamma_k| \frac{|\pi_\beta(z, \alpha_j)|}{|z - \alpha_k|} \frac{1}{|\pi'_\beta(\alpha_k, \alpha_j)|} \left(\frac{1 - |\alpha_k|}{|1 - \overline{\alpha_k} z|} \right)^m \frac{|f(z)|}{|f(\alpha_k)|} \leqslant$$

$$\leqslant c \sum_{k=1}^{+\infty} \exp \frac{\lambda}{(1 - |\alpha_k|)^{\alpha+1}} \frac{|\pi_\beta(z, \alpha_j)|}{|z - \alpha_k|} \exp \frac{M}{(1 - |\alpha_k|)^{\alpha+1}} \left(\frac{1 - |\alpha_k|}{|1 - \overline{\alpha_k} z|} \right)^m \frac{|f(z)|}{|f(\alpha_k)|}.$$

We estimate the factor $\dfrac{|\pi_\beta(z, \alpha_j)|}{|z - \alpha_k|}$. Using the well-known estimate for the Djrbashian product (see [13]):

$$\ln^+ |\pi_{\beta,k}(z, \alpha_j)| \leqslant c_\beta \sum_{k=1}^{+\infty} \left(\frac{1 - |\alpha_k|}{|1 - \overline{\alpha}_k z|} \right)^{\beta+2}, \tag{17}$$

we get:

$$\frac{|\pi_\beta(z, \alpha_j)|}{|z - \alpha_k|} = \frac{1}{|z - \alpha_k|} |\pi_{\beta,k}(z, \alpha_j)| \frac{|\alpha_k - z|}{|\alpha_k|} |\exp\left(-U_\beta(z, \alpha_k)\right)| \leqslant \tilde{c}_\beta \frac{|\pi_{\beta,k}(z, \alpha_j)|}{|1 - \overline{\alpha}_k z|},$$

$$\frac{|\pi_{\beta,k}(z, \alpha_j)|}{|1 - \overline{\alpha}_k z|} \leqslant \frac{\tilde{c}_\beta}{|1 - \overline{\alpha}_k z|} \exp \left(c_\beta \sum_{n=1}^{+\infty} \left(\frac{1 - |\alpha_n|}{|1 - \overline{\alpha}_n z|} \right)^{\beta+2} \right)$$

for all $\beta > \alpha - 1$.

Therefore

$$|\Psi(z)| \leqslant \exp \left(c_\beta \sum_{n=1}^{+\infty} \left(\frac{1 - |\alpha_n|}{|1 - \overline{\alpha}_n z|} \right)^{\beta+2} \right) \times |f(z)| \times \tilde{c}_\beta \cdot \sum_{k=1}^{+\infty} \exp \frac{\lambda + M}{(1 - |\alpha_k|)^{\alpha+1}} \cdot \frac{1}{|f(\alpha_k)|} \frac{(1 - |\alpha_k|)^m}{|1 - \overline{\alpha_k} z|^{m+1}}.$$

Now we consider the last factor in the product:we obtain the following estimate

$$\sum_{k=1}^{+\infty} \exp \frac{\lambda + M}{(1 - |\alpha_k|)^{\alpha+1}} \cdot \frac{1}{|f(\alpha_k)|} \frac{(1 - |\alpha_k|)^m}{|1 - \overline{\alpha_k} z|^{m+1}}.$$

We split the sum into n parts:

$$\sum_{s=1}^{n} \sum_{\alpha_k \in \Gamma_\delta(\theta_s)} \exp \frac{\lambda + M}{(1 - |\alpha_k|)^{\alpha+1}} \cdot \frac{1}{|f(\alpha_k)|} \frac{(1 - |\alpha_k|)^m}{|1 - \overline{\alpha_k} z|^{m+1}}.$$

Since $\{\alpha_k\} \subset \bigcup\limits_{s=1}^{n} \Gamma_\delta(\theta_s)$ for certain $0 < \delta < \dfrac{1}{\alpha+1}$, we can apply Lemma 1.3 for each part of the sum. Thus we have:

$$\sum_{s=1}^{n} \sum_{\alpha_k \in \Gamma_\delta(\theta_s)} \exp \frac{\lambda + M - C}{(1-|\alpha_k|)^{\alpha+1}} \cdot \frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k} z|^{m+1}}.$$

Choosing the positive constant C such that $\lambda + M - C < 0$, we obtain the following estimate:

$$\exp \frac{\lambda + M - C}{(1-|\alpha_k|)^{\alpha+1}} \leqslant 1,$$

for all $k = 1, 2, \ldots$.

Thus we have:

$$|\Psi(z)| \leqslant \exp\left(c_\beta \sum_{n=1}^{+\infty} \left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n z|}\right)^{\beta+2}\right) \times |f(z)| \times \widetilde{c}_\beta \cdot \sum_{k=1}^{+\infty} \frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k} z|^{m+1}}.$$

Taking into account the convergence of the series (16), we have:

$$\sum_{k=1}^{+\infty} \frac{(1-|\alpha_k|)^m}{|1-\overline{\alpha_k} z|^{m+1}} \leqslant \frac{c}{(1-|z|)^{m+1}} \sum_{k=1}^{+\infty} (1-|\alpha_k|)^m \leqslant \frac{c_1}{(1-|z|)^{m+1}}$$

for all $m > \alpha + 1$.

The estimate of function $|\Psi(z)|$ takes form:

$$|\Psi(z)| \leqslant \exp\left(c_\beta \sum_{n=1}^{+\infty} \left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n z|}\right)^{\beta+2}\right) \times |f(z)| \times \frac{c_2}{(1-|z|)^{m+1}}. \tag{18}$$

Now we show that $\Psi(z) \in S_\alpha^\infty$, in this notation $T(r, \Psi) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \ln^+ |\Psi(re^{i\theta})| d\theta \leqslant \dfrac{C}{(1-r)^\alpha}$, where $\alpha > 0$, $C > 0$.

Using (18), we get

$$T(r, \Psi) \leqslant c_\beta \cdot \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} \left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n re^{i\theta}|}\right)^{\beta+2} d\theta + \int_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta + 2\pi \ln \frac{c}{(1-r)^{m+1}}.$$

We estimate each of summands in this sum separately. As established in [11] (see also [14]),

$$\sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} \left(\frac{1-|\alpha_n|}{|1-\overline{\alpha}_n re^{i\theta}|}\right)^{\beta+2} d\theta \leqslant \frac{c}{(1-r)^\alpha}, \tag{19}$$

for all $\beta > \alpha - 1$.

Further, from (15) we have:

$$\int_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta \leqslant \sum_{s=1}^{n} \int_{-\pi}^{\pi} \frac{C}{|1-re^{i(\theta-\theta_s)}|^{\alpha+1}}.$$

Applying elementary estimate, we obtain:

$$\int_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta \leqslant \frac{c_3}{(1-r)^\alpha}. \tag{20}$$

Combining (19), (20), we obtain that function $\Psi(z)$ belongs to the class S_α^∞. This shows that ii) $\to$ i).

The proof of Theorem 1.1 is complete. □

Remark 2.1. Note that if $\alpha = 0$, then condition (5) is necessary, as established in [4].

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Об интерполяции в классах аналитических в круге функций со степенным ростом характеристики Р. Неванлинны

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В статье получено решение интерполяционной задачи в классе аналитических функций в единичном круге, характеристика Р. Неванлинны которых имеет степенной рост при приближении к единичной окружности, при условии, что узлы интерполяции принадлежат конечному числу углов Штольца.

Ключевые слова: интерполяция, аналитические функции, характеристика Р. Неванлинны, углы Штольца.