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Extremal Curves in the Conformal Space and in an Associated Bundle

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Time-like and null extremal curves are computed in the conformal space for bundles with fibers P_5 *and* $P_5 \wedge P_5$.

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Introduction

It is known that the movement of physical bodies in the Solar system and a light ray near the surface of the Sun is well modeled by time-like and null geodesic lines of the Lorentz 4-manifold. When Riemann curvature vanishes the Lorentzian manifold is locally isomorphic to the Minkowski space. Geodesic lines of a Lorentzian manifold are extremals of the functional of the arc length. In the case of the Minkowski space these extremals are straight lines, and they model free movement (no external forces) of a point or a light ray.

In [1] it is proposed to consider the 4-manifold with conformal connection and with the angular metric of the signature $(- + ++)$ as a model of space-time and to determine the dynamics in this model by the system of the Yang-Mills equations. The geometric structure of the manifold with conformal connection is much richer than the structure of the Lorentz manifold, so there is a hope to interpret in the new model not only the movement of such structureless object as a material point, but also the motion of structured objects such as elementary particles. In this interpretation it is naturally to follow the analogy with the already well-studied model, the Lorentzian manifold, and examine, first of all, extremal curves of a 4-manifolds with conformal connection. But, unlike in the Lorentzian manifold, extremal curves in a 4-manifold with conformal connection are very sophisticated structured objects even when the conformal curvature vanishes. Therefore, in this paper we study extremal curves in 4-manifolds with conformal connection of zero conformal curvature of maximum mobility, i.e. in the conformal space.

The model of the 4-dimensional conformal space is the quadric

$$(x,x) = 2x_0x_5 + \eta_{11}(x_1)^2 + \eta_{22}(x_2)^2 + \eta_{33}(x_3)^2 + \eta_{44}(x_4)^2 = 0 \tag{1}$$

in the projective space P_5 (here $\eta_{ii} = \pm 1$). In the tangent plane of the 4-manifold M determined by the quadric (1) there is the metric

$$\psi = \eta_{11}(dx_1)^2 + \eta_{22}(dx_2)^2 + \eta_{33}(dx_3)^2 + \eta_{44}(dx_4)^2 . \tag{2}$$

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Our ultimate goal is to study the model of space-time mentioned above, that is why we are interested in the case when the quadratic form (2) is the Minkowski metric. So we put

$$\eta_{11} = -1, \qquad \eta_{22} = \eta_{33} = \eta_{44} = 1. \tag{3}$$

It should be borne in mind that the quadric (1) allows many fiber bundles. The simplest case is the bundle with the projective space P_5 as a fiber. In this case we construct the fiber bundle as a Cartesian product $M \times P_5$, and the bundle projection is the projection from the product to the first factor. But we may consider an analogous trivial bundle if we take as a fiber the skew tensor product $P_5 \wedge P_5$ of two (or more) spaces P_5. Each curve $x(t)$ in M has a natural lifting to the fiber $P_5 \wedge P_5$ determined by the formula

$$[x(t), x'(t)] = \frac{1}{2}\left(x_\alpha(t)\, x'_\beta(t) - x_\beta(t)\, x'_\alpha(t)\right), \qquad \alpha, \beta \in \overline{0, 5}. \tag{4}$$

Any curve $x(t)$ in M has the arc length functional invariant under the conformal group $C(3,1)$ action. The conformal group $C(3,1)$ is the invariance group of equation (1) with the condition (3) in P_5. The conformal group also has a representation in the skew product $P_5 \wedge P_5$. But invariant arc length functionals for the curve $x(t)$ and the curve (4) are different, so the extremal curves corresponding to these functionals are also different.

In this article extremal curves of two kind are studied: time-like and null. Time-like curves are characterised by three functions of the arc length: m, c, and p, which for extremal curves in the bundle with fiber P_5 are expressed through the Weierstrass elliptic function and finally depend on four real parameters. In the bundle with fiber $P_5 \wedge P_5$ extremal time-like curves are characterised by conditions: $m = const$, $c = 0$, and p is an arbitrary function of the arc length.

Nonplane null curves are characterised by two functions of the arc length: f and g. For extremal curves in the bundle with fiber P_5 these functions are constant; there are no nonplane null extremals in the bundle with fiber $P_5 \wedge P_5$.

Plane null curves are characterised by one function of the arc length f and one discrete parameter $g = \pm 1$. Calculations for extremal curves in this case are so cumbersome that the authors develop a special computer program. It turns out that in the bundle with fiber P_5 the function $f(s)$ must satisfy the equation

$$12 f^{(4)} - 156 f f'' - 99 f'^2 + 144 f^3 - 56 f = \text{const}.$$

We could not find the general solution of the equation, but aside from the obvious particular solution $f = const$ we found two classes of solutions expressed in terms of the Weierstrass elliptic function: $f = 8\wp\left(s + g_1, \frac{7}{150}, g_3\right)$ and $f = \frac{5}{4}\wp\left(s + g_1, \frac{32}{3}, g_3\right)$, where g_1, g_3 are arbitrary constants. For a plane null extremal curve in the bundle with fiber $P_5 \wedge P_5$ the function $f = 0$.

It should be noted that in the study of extremal curves, as well as in finding solutions of the Yang-Mills equations in our papers [2] and [3], in many cases the Weierstrass elliptic function is involved. It is not by accident, but due to the fact that the Weierstrass function is double-periodic. This double periodicity is the most important property to applications. In particular, our solution of the Yang-Mills equations for the central-symmetric metric in [2], containing the Weierstrass function, was used by A. Trunev to study the structure of quarks, leptons, hadrons and atomic nuclei in [4–8].

Extremal curves on the quadric (1) were first studied in [9], but there were several computation mistakes. In the present work these mistakes are corrected, some formulations are refined, additionally the plane null extremal curve and extremal curves in the bundle with fiber $P_5 \wedge P_5$ are studied.

1. Preliminaries

If $x\,(x_0, x_1, x_2, x_3, x_4, x_5)$ and $z\,(z_0, z_1, z_2, z_3, z_4, z_5)$ are two points of P_5, their scalar product is defined as $(x, z) = x_0z_5 + x_5z_0 - x_1z_1 + x_2z_2 + x_3z_3 + x_4z_4$. To each point x we put into correspondence a frame $\{x, y_i, X\}$, $i \in 1, 2, 3, 4$, consisting of six linearly independent points of space P_5 satisfying the orthogonality conditions

$$(x, x) = (X, X) = (x, y_i) = (X, y_i) = 0, \;\; (x, X) = 1, \;\; (y_i, y_j) = \eta_{ij}, \tag{5}$$

where η_{ij} is the Minkowski tensor defined by the formula

$$(\eta_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The equations of infinitesimal transformations of the frame $\{x, y_i, X\}$ are as follows [10, p. 157]:

$$dx = \omega^k y_k + \omega_0^0 x, \quad dy_i = \omega_i^k y_k + \omega_i x + \omega_i^5 X, \quad dX = \omega_5^k y_k - \omega_0^0 X, \tag{6}$$

where pfaffian forms (infinitesimal coefficients) satisfy the equalities $\omega_i^5 + \eta_{ij}\omega^j = 0$, $\eta_{ij}\omega_5^j + \omega_i = 0$, $\eta_{ik}\omega_j^k + \eta_{jk}\omega_i^k = 0$. Hereinafter all indexes run over values $1, 2, 3, 4$; we assume summation over identical top and bottom indexes.

In this article we will use Lagrangians representable as functions of scalar products $\left(x^{(k)}, x^{(l)}\right)$, where $k, l \in \{0, 1, ..., 5\}$, and the symbol $x^{(k)}$ denotes the k-th derivative. If L is a Lagrangian, we denote by $L_{x^{(k)}}$ the value calculated in accordance with the formula

$$L_{x^{(k)}} \stackrel{def}{=} \sum_{l \neq k} \frac{\partial L}{\partial \left(x^{(k)}, x^{(l)}\right)} x^{(l)} + 2 \frac{\partial L}{\partial \left(x^{(k)}, x^{(k)}\right)} x^{(k)}.$$

In particular, if $L = \left(x^{(k)}, x^{(l)}\right)$, then

$$L_{x^{(p)}} = \begin{cases} 0, \text{ if } p \neq k, \; p \neq l; \\ x^{(l)}, \text{ if } p = k \neq l; \\ 2x^{(l)}, \text{ if } p = k = l. \end{cases} \tag{7}$$

2. Time-like curve

In this section we examine the movement of a frame $\{x, y_i, X\}$ along time-like curves, i.e. such curves where $(dx, dx) < 0$. Using formulas for movement of the canonic frame we will find the differential equation for extremal curves, i.e. the curves that are extremals of the functional $I = \int_{s_0}^{s_1} ds$, where ds is the arc length element of a time-like curve.

It was shown in [9] that for a time-like curve formulas (6) turn into

$$\begin{gathered} \bar{x}' = \overline{y_1}, \quad \overline{y_1}' = m\bar{x} + \overline{X}, \quad \overline{X}' = m\overline{y_1} - \overline{y_2}, \\ \overline{y_2}' = \overline{x} + c\overline{y_3}, \quad \overline{y_3}' = -c\overline{y_2} + p\overline{y_4}, \quad \overline{y_4}' = -p\overline{y_3}. \end{gathered} \tag{8}$$

Here $\bar{x} = \lambda x$ is an invariantly normalized point x, and the symbol $'$ denotes the derivative with respect to the invariant parameter s. First derivatives look like

$$\begin{gathered} \bar{x}' = \overline{y_1}, \;\; \overline{x}'' = m\bar{x} + \overline{X}, \;\; \overline{x}''' = m'\overline{x} + 2m\overline{y_1} - \overline{y_2}, \\ \overline{x}^{(4)} = \left(m'' + 2m^2 - 1\right) \overline{x} + 3m'\overline{y_1} - c\overline{y_3} + 2m\overline{X}, \\ \overline{x}^{(6)} = \left(m^{(4)} + 7m'^2 + 11mm'' - 3m + c^2 + 4m^3\right) \overline{x} + \left(5m''' + 20mm'\right) \overline{y_1} + \\ + \left(3cc' - 7m'\right) \overline{y_2} + c^3 + cp^2 - 2cm - c''\overline{y_3} - \left(cp' + 2c'p\right) \overline{y_4} + \left(4m^2 + 9m'' - 1\right) \overline{X}. \end{gathered} \tag{9}$$

Let us denote by $A_k(x)$ the following Gram determinant:

$$A_k(x) = \begin{vmatrix} (x,x) & (x,dx) & ... & (x,d^k x) \\ (dx,x) & (dx,dx) & ... & (dx,d^k x) \\ ... & ... & ... & ... \\ (d^k x,x) & (d^k x,dx) & ... & (d^k x,d^k x) \end{vmatrix}.$$

If λ is a function, as it is known, the following property of homogeneity holds: $A_k(\lambda x) = \lambda^{2k+2} A_k(x)$. In our case $(x,x)=0$, consequently, $(dx,x)=(x,dx)=0$ and the determinants $A_0(x)$ and $A_1(x)$ vanish. Four determinants $A_k(x)$ at $k=2,3,4,5$ give the possibility to obtain invariant formulas for invariant of the arc length for time-like and null curves. For this purpose it is necessary to make all possible algebraic combinations of these determinants that are invariant under normalization and of the first degree of homogeneity with respect to the differential. Notice that every summand of the determinant $A_k(x)$ has the same sum of indexes at the differentials ocurring in this summand as factors, and this sum is equal to $k(k+1)$. So, to the determinant $A_k(x)$ corresponds a pair of natural numbers $(2k+2,\ k(k+1))$, first of which is the degree of homogeneity with respect to the variable x, and the second is the degree of homogeneity of the sum of indexes at the differentials. We call this pair *the homogeneity index*. In particular

$$A_2 \longrightarrow (6,6),\ A_3 \longrightarrow (8,12),\ A_4 \longrightarrow (10,20),\ A_5 \longrightarrow (12,30). \tag{10}$$

Let us make such a combination of modules $|A_2|^p \cdot |A_3|^q$ with unknown real powers p and q that it would have the homogeneity index $(0,1)$. According to the first two correspondences (10), p and q must satisfy the system of linear equations [11, p. 43]

$$6p+8q=0, \qquad 6p+12q=1. \tag{11}$$

Hence $p=-\dfrac{1}{3}$, $q=\dfrac{1}{4}$. From here it follows that a candidate for the invariant differential of the arc length should look like

$$ds = \frac{|A_3|^{\frac{1}{4}}}{|A_2|^{\frac{1}{3}}}. \tag{12}$$

From formulas (9) and the orthogonality conditions (5) we have

$$A_3(\overline{x}) = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2m \\ 1 & 0 & 2m & m' \\ 0 & -2m & m' & 1-4m^2 \end{vmatrix} ds^{12} = ds^{12}.$$

Similarly, $A_2(\overline{x}) = ds^6$, therefore (12) is indeed the invariant differential.

Now we will look for the extremals of the functional $I = \displaystyle\int_{s_0}^{s_1} ds = \int_{t_0}^{t_1} L(t)\,dt$. According to the reasoning above, the Lagrangian L is

$$L(t) = \frac{(\Delta_3(t))^{\frac{1}{4}}}{(\Delta_2(t))^{\frac{1}{3}}}, \tag{13}$$

where

$$\Delta_3(t) = \begin{vmatrix} (x,x) & (x,\dot{x}) & (x,\ddot{x}) & (x,\dddot{x}) \\ (\dot{x},x) & (\dot{x},\dot{x}) & (\dot{x},\ddot{x}) & (\dot{x},\dddot{x}) \\ (\ddot{x},x) & (\ddot{x},\dot{x}) & (\ddot{x},\ddot{x}) & (\ddot{x},\dddot{x}) \\ (\dddot{x},x) & (\dddot{x},\dot{x}) & (\dddot{x},\ddot{x}) & (\dddot{x},\dddot{x}) \end{vmatrix},\quad \Delta_2(t) = \begin{vmatrix} (x,x) & (x,\dot{x}) & (x,\ddot{x}) \\ (\dot{x},x) & (\dot{x},\dot{x}) & (\dot{x},\ddot{x}) \\ (\ddot{x},x) & (\ddot{x},\dot{x}) & (\ddot{x},\ddot{x}) \end{vmatrix}$$

(here the symbol $\dot{x}$ denotes the derivative with respect to the parameter t).

As we have an equality constraint $(x,x)=0$, we need the Euler-Lagrange equation for the Lagrangian

$$L^*(t) = L(t) + \lambda \cdot (x,x), \tag{14}$$

where $\lambda(t)$ is an arbitrary function. Extremal curves must satisfy the Euler-Lagrange equation

$$\frac{d^3}{dt^3}L^*_{\dddot{x}} - \frac{d^2}{dt^2}L^*_{\ddot{x}} + \frac{d}{dt}L^*_{\dot{x}} - L^*_{x} = 0. \tag{15}$$

Using (7), we will have

$$(\Delta_3(t))_x = \begin{vmatrix} 2x & (x,\dot{x}) & (x,\ddot{x}) & (x,\dddot{x}) \\ \dot{x} & (\dot{x},\dot{x}) & (\dot{x},\ddot{x}) & (\dot{x},\dddot{x}) \\ \ddot{x} & (\ddot{x},\dot{x}) & (\ddot{x},\ddot{x}) & (\ddot{x},\dddot{x}) \\ \dddot{x} & (\dddot{x},\dot{x}) & (\dddot{x},\ddot{x}) & (\dddot{x},\dddot{x}) \end{vmatrix} + \begin{vmatrix} (x,x) & \dot{x} & (x,\ddot{x}) & (x,\dddot{x}) \\ (\dot{x},x) & 0 & (\dot{x},\ddot{x}) & (\dot{x},\dddot{x}) \\ (\ddot{x},x) & 0 & (\ddot{x},\ddot{x}) & (\ddot{x},\dddot{x}) \\ (\dddot{x},x) & 0 & (\dddot{x},\ddot{x}) & (\dddot{x},\dddot{x}) \end{vmatrix} +$$

$$+ \begin{vmatrix} (x,x) & (x,\dot{x}) & \ddot{x} & (x,\dddot{x}) \\ (\dot{x},x) & (\dot{x},\dot{x}) & 0 & (\dot{x},\dddot{x}) \\ (\ddot{x},x) & (\ddot{x},\dot{x}) & 0 & (\ddot{x},\dddot{x}) \\ (\dddot{x},x) & (\dddot{x},\dot{x}) & 0 & (\dddot{x},\dddot{x}) \end{vmatrix} + \begin{vmatrix} (x,x) & (x,\dot{x}) & (x,\ddot{x}) & \dddot{x} \\ (\dot{x},x) & (\dot{x},\dot{x}) & (\dot{x},\ddot{x}) & 0 \\ (\ddot{x},x) & (\ddot{x},\dot{x}) & (\ddot{x},\ddot{x}) & 0 \\ (\dddot{x},x) & (\dddot{x},\dot{x}) & (\dddot{x},\ddot{x}) & 0 \end{vmatrix}.$$

Since the transposition does not change the determinant of a matrix,

$$(\Delta_3(t))_x = 2\begin{vmatrix} x & (x,\dot{x}) & (x,\ddot{x}) & (x,\dddot{x}) \\ \dot{x} & (\dot{x},\dot{x}) & (\dot{x},\ddot{x}) & (\dot{x},\dddot{x}) \\ \ddot{x} & (\ddot{x},\dot{x}) & (\ddot{x},\ddot{x}) & (\ddot{x},\dddot{x}) \\ \dddot{x} & (\dddot{x},\dot{x}) & (\dddot{x},\ddot{x}) & (\dddot{x},\dddot{x}) \end{vmatrix}.$$

Thus, we have obtained **the rule for finding values $(\Delta_p(t))_{x^{(k)}}$: it is necessary to insert the column** $2\begin{pmatrix} x \\ \dot{x} \\ \dots \\ x^{(k)} \end{pmatrix}$ **in $\Delta_p(t)$ instead of the $(k+1)$th column.**

Now we turn to the invariant parameter s

$$(\Delta_3)_{\overline{x}} = 2\begin{vmatrix} \overline{x} & 0 & 1 & 0 \\ \overline{x}' & -1 & 0 & -2m \\ \overline{x}'' & 0 & 2m & m' \\ \overline{x}''' & -2m & m' & 1-4m^2 \end{vmatrix} = -2m'\overline{x}''' + 2\overline{x}'' + 4mm'\overline{x}' + \left(2m'^2 - 4m\right)\overline{x}.$$

By similar calculations,

$$(\Delta_2)_{\overline{x}} = 2\begin{vmatrix} \overline{x} & 0 & 1 \\ \overline{x}' & -1 & 0 \\ \overline{x}'' & 0 & 2m \end{vmatrix} = 2\overline{x}'' - 4m\overline{x}.$$

From (13) and (14) it follows that $L^*_{\overline{x}} = \frac{1}{4}(\Delta_3)_{\overline{x}} - \frac{1}{3}(\Delta_2)_{\overline{x}} + 2\lambda\overline{x}$, consequently,

$$L^*_{\overline{x}} = -\frac{1}{2}m'\overline{x}''' - \frac{1}{6}\overline{x}'' + mm'\overline{x}' + \left(\frac{1}{2}m'^2 + \frac{1}{3}m\right)\overline{x} + 2\lambda\overline{x}. \tag{16}$$

Similar computations show that

$$\begin{gathered} \frac{d}{ds}L^*_{\overline{x}'} = -m\overline{x}^{(4)} - m'\overline{x}''' + \left(2m^2 + \frac{1}{6}\right)\overline{x}'' + 5mm'\overline{x}' + \left(mm'' + m'^2\right)\overline{x}, \\ \frac{d^2}{ds^2}L^*_{\overline{x}''} = -\frac{1}{6}\overline{x}'', \quad \frac{d^3}{ds^3}L^*_{\overline{x}'''} = \frac{1}{2}\overline{x}^{(6)} - m\overline{x}^{(4)} - \frac{7}{2}m'\overline{x}''' - \frac{9}{2}m''\overline{x}'' - \frac{5}{2}m'''\overline{x}' - \frac{1}{2}m^{(4)}\overline{x}. \end{gathered} \tag{17}$$

Substitute expressions from formulas (16) and (17) into the Euler-Lagrange equation (15)

$$\begin{gathered}\overline{x}^{(6)}-4m\overline{x}^{(4)}-8m'\overline{x}'''+\left(4m^2+1-9m''\right)\overline{x}''+\left(8mm'-5m'''\right)\overline{x}'+\\+\left(2mm''+m'^2-m^{(4)}-\frac{2}{3}m-4\lambda\right)\overline{x}=0.\end{gathered}\tag{18}$$

Using (9), we obtain

$$\left(c^2+\frac{4}{3}m-4\lambda\right)\overline{x}+\left(3cc'+m'\right)\overline{y_2}+\left(c^3-c''+cp^2+2cm\right)\overline{y_3}-\left(2c'p+cp'\right)\overline{y_4}=0.$$

This results in the system of equations

$$c^2+\frac{4}{3}m=4\lambda,\quad 3cc'+m'=0,\quad c^3-c''+cp^2+2cm=0,\quad 2c'p+cp'=0.\tag{19}$$

Integrate the 2nd and the 4th equations of this system:

$$3c^2+2m=3A,\qquad c^2p=B,\qquad A,B=const.\tag{20}$$

Then we reduce the 3rd equation (19) to the variable c. It leads to $c''+2c^3-\dfrac{B^2}{c^3}-3Ac=0$. In this equation there is no independent variable s, therefore it admits an order reduction: $c'^2=-c^4+3Ac^2-\dfrac{B^2}{c^2}+D$, where $D=const$. Denote $q\stackrel{def}{=}A-c^2=\dfrac{2}{3}m$, the last equation turns into

$$q'^2=4q^3-g_2q-g_3,\tag{21}$$

where $g_2=12A^2+4D$, $g_3=4B^2-8A^3-4AD$. Thus, $q=\dfrac{2}{3}m$ is the elliptic Weierstrass function $q=\wp\left(s+g_1,g_2,g_3\right)$, where g_1 is an arbitrary constant of integration of equation (21). Together with (20) it gives the complete solution of the system (19).

Triple differentiation of the equality (21) and $q=\dfrac{2}{3}m$ results in $m'''=8mm'$, $m^{(4)}=8m'^2+8mm''$. Substitution of these expressions and the first equality of (19) in (18) gives the **final variant of the Euler-Lagrange equation for a time-like curve:**

$$\begin{gathered}\overline{x}^{(6)}-4m\overline{x}^{(4)}-8m'\overline{x}'''+\left(4m^2+1-9m''\right)\overline{x}''-32mm'\overline{x}'-\\-\left(6mm''+7m'^2+\frac{4}{3}m+A\right)\overline{x}=0,\end{gathered}\tag{22}$$

$m=\dfrac{3}{2}\wp\left(s+g_1,g_2,g_3\right)$, where A,g_1,g_2,g_3 are arbitrary constants.

Till now we have examined a space with conformal connection where fiber is the projective space P_5. Now we find extremal curves in the case when the fiber is $P_5\wedge P_5$. The scalar product in the space $P_5\wedge P_5$ is defined as follows: if $v=[a,b]$, $w=[c,d]$; $a,b\in P_5$; $v,w\in P_5\wedge P_5$, then

$$(v,w)\stackrel{def}{=}(a,c)(b,d)-(a,d)(b,c).\tag{23}$$

Denote $u\stackrel{def}{=}[x,x']$. From formulas (8) we obtain

$$\begin{gathered}\overline{u}=\left[\overline{x},\overline{y_1}\right],\qquad \overline{u}'=\left[\overline{x},\overline{X}\right],\qquad \overline{u}''=\left[\overline{y_1},\overline{X}\right]+m\left[\overline{x},\overline{y_1}\right]-\left[\overline{x},\overline{y_2}\right],\\ \overline{u}'''=2m\left[\overline{x},\overline{X}\right]-c\left[\overline{x},\overline{y_3}\right]-2\left[\overline{y_1},\overline{y_2}\right]+m'\left[\overline{x},\overline{y_1}\right].\end{gathered}\tag{24}$$

Further, using (23), we find scalar products and consider the Gram determinants. The first two nonzero determinants are

$$\Gamma_2(s)=\begin{vmatrix}(\overline{u},\overline{u}) & (\overline{u},\overline{u}') & (\overline{u},\overline{u}'')\\ (\overline{u},\overline{u}') & (\overline{u}',\overline{u}') & (\overline{u}',\overline{u}'')\\ (\overline{u},\overline{u}'') & (\overline{u}',\overline{u}'') & (\overline{u}'',\overline{u}'')\end{vmatrix}=\begin{vmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 2m\end{vmatrix}=1,\tag{25}$$

and $\Gamma_3(s) = -4$. The Lagrangian, similar to (14), is $L^*(t) = \dfrac{(-\Gamma_3(t))^{\frac{1}{4}}}{\sqrt{2}\,(\Gamma_2(t))^{\frac{1}{3}}} + \lambda \cdot (u,u)$.

Computations similar to those with the variation with respect to x lead to the Euler-Lagrange equation

$$\begin{aligned}\overline{u}^{(6)} - 4m\overline{u}^{(4)} - 8m'\overline{u}''' + \left(4m^2 - 4 - 9m''\right)\overline{u}'' + \left(8mm' - 5m'''\right)\overline{u}' + \\ + \left(-m^{(4)} + m'^2 + 2mm'' + \frac{8}{3}m + 32\lambda\right)\overline{u} = 0.\end{aligned} \tag{26}$$

Substitute into (26) expressions from (24) and from

$$\begin{aligned}
\overline{u}^{(4)} &= \left(2m^2 + 2 + m''\right)[\overline{x}, \overline{y_1}] + \left(c^2 - 4m\right)[\overline{x}, \overline{y_2}] - c'[\overline{x}, \overline{y_3}] - \\
&\quad - cp[\overline{x}, \overline{y_4}] + 3m'\left[\overline{x}, \overline{X}\right] - 3c[\overline{y_1}, \overline{y_3}] + 2m\left[\overline{y_1}, \overline{X}\right] + 2\left[\overline{y_2}, \overline{X}\right], \\
\overline{u}^{(6)} &= \left(m^{(4)} + 7m'^2 + 11mm'' + 4m^3 + 12m - 4c^2\right)[\overline{x}, \overline{y_1}] + \left(9m''' + 20mm'\right)\left[\overline{x}, \overline{X}\right] + \\
&\quad + \left(3c'^2 + 4cc'' - 11m'' - c^4 + 7c^2m - c^2p^2 - 12m^2 - 4\right)[\overline{x}, \overline{y_2}] + \\
&\quad + \left(6c^2c' - 14cm' + 3c^2c' - 11c'm - c''' + 3c'p^2 + 3cpp'\right)[\overline{x}, \overline{y_3}] + \\
&\quad + \left(c^3p - 11cmp - 3c''p + cp^3 - 3c'p' - cp''\right)[\overline{x}, \overline{y_4}] + \left(15cc' - 20m'\right)[\overline{y_1}, \overline{y_2}] + \\
&\quad + \left(5c^3 - 20cm - 5c'' + 5cp^2\right)[\overline{y_1}, \overline{y_3}] - \left(10c'p + 5cp'\right)[\overline{y_1}, \overline{y_4}] + 5c[\overline{y_2}, \overline{y_3}] + \\
&\quad + \left(4m^2 + 9m'' + 4\right)\left[\overline{y_1}, \overline{X}\right] + \left(8m - 9c^2\right)\left[\overline{y_2}, \overline{X}\right] + 9c'\left[\overline{y_3}, \overline{X}\right] + 9cp\left[\overline{y_4}, \overline{X}\right].
\end{aligned}$$

Having equated to zero the coefficients at $[\overline{y_2}, \overline{y_3}]$, $[\overline{y_1}, \overline{y_2}]$ and $[\overline{x}, \overline{y_1}]$, we find $c = 0$, $m' = 0$, $\lambda = -\dfrac{1}{12}m$. Under these conditions all other coefficients vanish for an arbitrary function p. We substitute $c = 0$, $m' = 0$ and $\lambda = -\dfrac{1}{12}m$ into (26) and get the required **equation of extremal time-like curves in the case with the variation with respect to** $\overline{u} = [\overline{x}, \overline{x}']$

$$\overline{u}^{(6)} - 4m\overline{u}^{(4)} + \left(4m^2 - 4\right)\overline{u}'' = 0. \tag{27}$$

3. Nonplane null curve

Further we examine the movement of the frame $\{x, y_i, X\}$ along null curves, i.e. such curves that $(dx, dx) = 0$. To simplify computations, we change the Minkowski tensor

$$(\eta_{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case $(y_1, y_1) = (y_2, y_2) = 0$, $(y_1, y_2) = 1$. Other conditions of orthogonality (5) remain unchanged.

Formulas of infinitesimal transformations of the canonic frame for a null curve are [9]

$$\begin{aligned}&\overline{x}' = \overline{y_0}, \quad \overline{y_0}' = f\overline{x} + \overline{y_2}, \quad \overline{y_2}' = \frac{3}{2}f\overline{y_0} - \overline{y_1}, \\ &\overline{y_1}' = -\frac{3}{2}f\overline{y_2} + g\overline{x} - \overline{X}, \quad \overline{X}' = -g\overline{y_0} - f\overline{y_1} - \overline{y_3}, \quad \overline{y_3}' = \overline{x}.\end{aligned} \tag{28}$$

In the case of a null curve the only nonzero Gram determinants are Δ_4

$$\Delta_4(s) = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 5f \\ 0 & -1 & 0 & -5f & -\frac{5}{2}f' \\ 1 & 0 & 5f & -\frac{5}{2}f' & 2f'' + 21f^2 - 2g \end{vmatrix} = 1, \tag{29}$$

and $\Delta_5(s) = 1$.

Also, in the case of a null curve there are two equality constraints: $(x, x) = 0$ and $(x', x') = 0$, the Lagrangian, similar to (14), turns into

$$L^*(t) = \frac{(\Delta_5(t))^{\frac{1}{6}}}{(\Delta_4(t))^{\frac{1}{5}}} + \lambda \cdot (x, x) + \mu \cdot (x', x'),$$

where $\lambda(t)$ and $\mu(t)$ are arbitrary functions. Extremal curves must satisfy the Euler-Lagrange equation

$$\frac{d^5}{ds^5} L^*_{\overline{x}^{(5)}} - \frac{d^4}{ds^4} L^*_{\overline{x}^{(4)}} + \frac{d^3}{ds^3} L^*_{\overline{x}'''} - \frac{d^2}{ds^2} L^*_{\overline{x}''} + \frac{d}{ds} L^*_{\overline{x}'} - L^*_{\overline{x}} = 0.$$

The computations appear to be much more cumbersome than in the case of a time-like curve, we omit them. The final result is

Claim 1.

A nonplane null curve is extremal iff the invariants f and g are constant. The equation for extremals is

$$\overline{x}^{(6)} - 5f\overline{x}^{(4)} + 2\left(2f^2 + g\right)\overline{x}'' + \overline{x} = 0.$$

Variation of the Euler-Lagrange equation with respect to u instead of x leads to a contradictory system, i.e. **with variation with respect to $u = [x, x']$ there are no extremal nonplane null curves.**

4. Plane null curve

Formulas of infinitesimal transformations of the canonic frame for a plane null curve are similar to (28)

$$\begin{aligned} &\overline{x}' = \overline{y_1}, \quad \overline{y_1}' = f\overline{x} + \overline{y_3}, \quad \overline{y_3}' = \frac{3}{2}f\overline{y_1} - \overline{y_2}, \\ &\overline{y_2}' = -\frac{3}{2}f\overline{y_3} + g\overline{x} - \overline{X}, \quad \overline{X}' = -g\overline{y_1} - f\overline{y_2}, \quad \overline{y_4}' = 0, \end{aligned} \tag{30}$$

where $g = \pm 1$, $f(s)$ is a function. However, they are not a special case of (28), therefore the study of plane null curves demands a separate consideration. Formulas (30) define a curve

$$\overline{x}^{(5)} - 5f\overline{x}''' - \frac{15}{2}f'\overline{x}'' + \left(4f^2 + 2g - \frac{9}{2}f''\right)\overline{x}' + (4ff' - f''')\overline{x} = 0. \tag{31}$$

The problem is to find conditions on function $f(s)$ such that this curve will be extremal, i.e. satisfy the Euler-Lagrange equation.

The first nonzero Gram determinant $\Delta_4(s)$ coincides with (29), however, $\Delta_5(s)$ and all determinants of greater order vanish. Hence, it is impossible to construct the required Lagrangian only using determinants $\Delta_k(s)$. Therefore we construct another Gram determinant from scalar products $u = [x, x']$ and its derivatives $u', u''...$, though the variation will still be with respect to x.

Among determinants $\Gamma_k(s)$ of the form (25) the first nonzero one is $\Gamma_6(s) = 5^7$. The homogeneity index of $\Gamma_k(s)$ is $\Gamma_k(s)(4k+4;(k+1)(k+2))$, hence $\Gamma_6(28;56)$. The determinant Δ_4 has the homogeneity index $\Delta_4(10;20)$. These indexes are proportional, therefore it is impossible to construct the required Lagrangian L with the homogeneity index $L(0;1)$ using Γ_6 and Δ_4. A suitable determinant is $\dfrac{\Gamma_7(s)}{5^8}$, which is equal to

$$\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -14f \\
0 & 0 & 0 & 0 & 1 & 0 & 14f & 35f' \\
0 & 0 & 0 & -1 & 0 & -14f & -21f' & a \\
0 & 0 & 1 & 0 & 14f & 7f' & 21f''+147f^2+6g & 28f'''+441ff' \\
0 & -1 & 0 & -14f & 7f' & -14f''-147f^2-6g & -7f'''-147ff' & b \\
1 & 0 & 14f & -21f' & 21f''+147f^2+6g & -7f'''-147ff' & c & \dfrac{c'}{2} \\
0 & -14f & 35f' & a & 28f'''+441ff' & b & \dfrac{c'}{2} & d
\end{vmatrix},$$

here

$$\begin{aligned}
&a = -42f''-147f^2-6g, \quad b = -7f^{(4)}-147ff''-147f'^2-c,\\
&c = 1408f^3 + 154fg + 549ff'' + 3f^{(4)} - \frac{1863}{4}f'^2,\\
&d = -13013f^4 - 2548f^2g - 52 + 15785ff'^2 - 1764f''^2 - 12628f^2f'' - 504f''g+\\
&+1960f'f''' - 280ff^{(4)}.
\end{aligned}$$

Computations show that $\Gamma_7 = -5^8 \cdot 16$. Its homogeneity index is $\Gamma_7\,(32;72)$. Having solved the system similar to (11), $32p+10q=0$, $72p+20q=1$, we obtain $p=\dfrac{1}{8}$, $q=-\dfrac{2}{5}$, and the required Lagrangian has the form

$$L^* = \frac{1}{5\sqrt{2}}\frac{(-\Gamma_7)^{\frac{1}{8}}}{(\Delta_4)^{\frac{2}{5}}} + \lambda\cdot(x,x) + \mu\cdot(x',x').$$

where $\lambda\,(t)$ and $\mu\,(t)$ are arbitrary functions. The values that are necessary for the Euler-Lagrange equation are

$$L^*_{\overline{x}^{(k)}} = -\frac{\frac{1}{5^7}\,(\Gamma_7)_{\overline{x}^{(k)}}}{640} - \frac{2}{5}\,(\Delta_4)_{\overline{x}^{(k)}} + \lambda\,(\overline{x},\overline{x})_{\overline{x}^{(k)}} + \mu(\overline{x}',\overline{x}')_{\overline{x}^{(k)}}. \tag{32}$$

The Euler-Lagrange equation for a null plane curve is

$$\frac{d^8}{ds^8}L^*_{\overline{x}^{(8)}} - \frac{d^7}{ds^7}L^*_{\overline{x}^{(7)}} + \dots + L^*_{\overline{x}} = 0.$$

The main computation difficulty arises when finding the values $(\Gamma_7)_{\overline{x}^{(k)}}$ in (32). As Γ_7 depends on scalar products $\left(u^{(k)},u^{(l)}\right)$ and we need to variate with respect to x and not with respect to u, it is impossible to use the rule formulated in the section about time-like curves. It is necessary to use formula (7) directly. To carry out tedious computations of $\left(u^{(k)},u^{(l)}\right)$ it was necessary to develop a special computer program calculating $\left(u^{(k)},u^{(l)}\right)_{\overline{x}^{(p)}}$. This helped to overcome all computational difficulties. The result is that the Euler-Lagrange equation leads to the only condition on the function f: it must satisfy the differential equation

$$12f^{(4)} - 156ff'' - 99f'^2 + 144f^3 - 56f = C, \tag{33}$$

where $C = const$. The complete solution of this equation was not found, however, except obvious $f = const$, it is possible to specify two series of solutions expressed by means of the elliptic Weierstrass function.

Let us denote $f = 8q$. The equation turns into

$$q^{(4)} - 104qq'' - 66q'^2 + 768q^3 - \frac{14}{3}q = const.$$

Direct substitution proves that a solution of this equation satisfies the equation $q' = \sqrt{4q^3 - \frac{7}{150}q - g_3}$, i.e. the Weierstrass function $q = \wp\left(s + g_1, \frac{7}{150}, g_3\right)$, where g_1, g_3 are arbitrary constants. Thus, the function $f = 8\wp\left(s + g_1, \frac{7}{150}, g_3\right)$ is a solution of (33). In a similar way, the function $f = \frac{5}{4}\wp\left(s + g_1, \frac{32}{3}, g_3\right)$ is also a solution of (33). We have proved

Claim 2

The plane null curve (31), $g = \pm 1$, is extremal iff the function f satisfies the equation (33). Functions $f = const$, $f = 8\wp\left(s + g_1, \frac{7}{150}, g_3\right)$ and $f = \frac{5}{4}\wp\left(s + g_1, \frac{32}{3}, g_3\right)$ are particular solutions of the equation.

While variating with respect to $u = [x, x']$, there are three equality constraints $(u, u) = 0$, $(u', u') = 0$ and $(u'', u'') = 0$, therefore the Lagrangian looks like

$$L^* = \frac{1}{\sqrt{2}} \frac{(-\Gamma_7)^{\frac{1}{8}}}{(\Gamma_6)^{\frac{1}{7}}} + \lambda \cdot (u, u) + \mu \cdot (u', u') + \nu \cdot (u'', u''),$$

where $\lambda(t)$, $\mu(t)$ and $\nu(t)$ are arbitrary functions. It appears that a curve is extremal only if $f = 0$. **The equation of an extremal plane null curve with variation with respect to $u = [x, x']$ is**

$$\overline{u}^{(9)} - 6g\overline{u}^{(5)} - 16\overline{u}' = 0, \qquad g = \pm 1.$$

Conclusion

By calculating some of the possible extremal curves in the conformal space, we have shown what kind of opportunities are opened for the interpretation of free movement of any particles. Even the quarks are no exception. But further work is needed. In particular, it is necessary to find out which extremals can entirely be located in a bounded part of space.

The biggest surprise for the authors is an abundance of null extremals. It is not clear what kind of light-like particle would move along a nonplane null extremal curve. The photon motion can be naturally associated with the plane null extremal curve. But it turns out that a plane null extremal curve depends on too many parameters: one discrete $g = \pm 1$, which corresponds to spin, and five continuous parameters. Most likely it is due to the fact that the plane null extremals model not only the movement of photon, but also some other light-like particles, such as gluons.

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Экстремальные кривые конформного пространства и одного из связанных с ним расслоений

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Вычислены экстремальные временеподобные и изотропные кривые конформного пространства для расслоений со слоями P_5 и $P_5 \wedge P_5$.

Ключевые слова: многообразие конформной связности, конформное пространство, уравнение Эйлера–Лагранжа, экстремальные кривые, эллиптическая функция Вейерштрасса.